Problem of the Week, Fall 2006 14 September 2006
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Problem 2

There is a natural number with the last digit of 4 (e.g. 1004). When you move the digit 4 in front of the number (e.g. 4100), the resulting number becomes the quadruple of the original number. Obviously, 1004 is not such a number. Find a natural number with the above characteristic, or prove that such numbers do not exist logically or empirically.

Answer: 102564.

Solution: Stating the requirement as an equation, we get:

$$\frac{(x-4)}{10}+4\times 10^{y-1}=4x$$

where x is the original number and y is the number of digits in the original natural number.

The equation solves for x as:

$$x=\frac{4\left(10^{y}-1\right)}{39}$$

and, of course, integer solutions to this equation (for x), when integers are substituted for y, are solutions to the problem. These integer solutions occur at $y = 6$, so that $x = 102564$, $y = 12$, where $x = 102564102564$, and every other y which is a multiple of 6, where x in each case is 102564 repeated a number of times equal to the particular multiple of 6 that y is.

(Challenge Problem (try it if you dare): What if we change the condition from quadruple to double? The last digit of the original number is still 4. Is it possible or not?)

In order to be possible, the following equation must be satisfied with integer values of x and y.

$$x=\frac{4\left(10^{y}-1\right)}{19}$$

This requires 10^y-1 = 999…999 to be evenly divisible by 19, which occurs when $y = 19$.
Here $x=\frac{4\left(10^{19}-1\right)}{19}=210{,}526{,}315{,}789{,}473{,}684$. (Ouch, manual multiplication!)