Problem of the Week, Fall 2006 28 September 2006

Circle One: Cadet Faculty Student (non USMA) Faculty (non USMA) Other

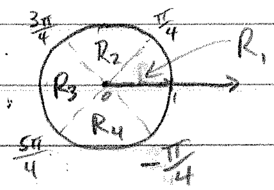
Last Name (Please Print): GRAVES

Problem 4

This week we have a MA 205 Integral Calculus special.

Three infinitely long circular cylinders each with unit radius have their axis along the x, y, and z-axes. Determine the volume of the region common to all three cylinders. (i.e. Find the volume common to $\{y^2+z^2 \le 1\}$, $\{z^2+x^2 \le 1\}$, $\{x^2+y^2 \le 1\}$.)

CONSIDER THE CYLINDER $x^2+y^2 \le 1$. IN POLAR COORDINATES, IT IS $r \le 1$. FROM ABOVE, WE DIVIDE THE CIRCLE INTO 4 REGIONS, R_1, R_2, R_3, R_4.



ABOVE AND BELOW R_1 AND R_3, THE VOLUME OF INTERESTED IS BOUNDED BY $x^2+z^2=1$, OR $-\sqrt{1-x^2} \le z \le \sqrt{1-x^2}$. NOTE $\sqrt{1-x^2} = \sqrt{1-r^2\cos^2\theta}$

ABOVE AND BELOW R_2 AND R_4, THE VOLUME OF INTEREST IS BOUNDED BY $z^2+y^2=1$, OR $-\sqrt{1-y^2} \le z \le \sqrt{1-y^2}$. NOTE $\sqrt{1-y^2} = \sqrt{1-r^2\sin^2\theta}$

SINCE THE VOLUME IS SYMMETRIC ABOUT THE PLANE $x=-y$, WE CAN WRITE

$$V = 2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\int_0^1 \left[\sqrt{1-r^2\cos^2\theta} - \left(-\sqrt{1-r^2\cos^2\theta}\right)\right] r\,dr\,d\theta + 2\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\int_0^1 \left[\sqrt{1-r^2\sin^2\theta} - \left(-\sqrt{1-r^2\sin^2\theta}\right)\right] r\,dr\,d\theta$$

$$= 4\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\int_0^1 \sqrt{1-r^2\cos^2\theta}\, r\,dr\,d\theta + 4\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\int_0^1 \sqrt{1-r^2\sin^2\theta}\, r\,dr\,d\theta$$

EVALUATING THIS INTEGRAL USING MATHEMATICA, WE FIND

$$V = 8(2-\sqrt{2})$$