Problem of the Week, AY 2007 5 April 2007

Circle One: Cadet Faculty Student (non USMA) Faculty (non USMA) Other

Last Name (Please Print):

Problem 20

The two lines in a 3 dimensional space are defined by the following parameterized vector functions:

Line 1: $\vec{r}_1(t) = < t+1,\ 2t+4,\ -3t+5 >$

Line 2: $\vec{r}_2(t) = < 4t-12,\ -t+8,\ t+17 >$

where $t \in \Re$.

What is the equation of the smallest sphere which is tangent to both lines?

Let the lines be l & m. Let P and Q be the points on the lines that touch the sphere. $\overline{PQ}$ is the diameter of the sphere.

l: $\vec{a} + t\vec{v}$, m: $\vec{b} + t\vec{w}$, Since the lines are not parallel, $\vec{v}$ & $\vec{w}$ are linearly independent. Since $\overline{PQ}$ is perpendicular to both lines, $\vec{PQ} = \rho \vec{v} \times \vec{w}$ for some constant ρ. Let $P = \vec{a} + \sigma v$ and $Q = \vec{b} + \tau \vec{w}$. Then $\vec{PQ} = \vec{b} - \vec{a} - \sigma\vec{v} + \tau\vec{w}$

$\vec{a} - \vec{b} = -\rho(\vec{v} \times \vec{w}) - \sigma\vec{v} + \tau\vec{w}$. So we can calculate ρ, σ, τ by expressing $\vec{a} - \vec{b}$ interms of $\vec{v} \times \vec{w}$, $\vec{v}$, and $\vec{w}$

Center of the shpere: $\vec{a} + \sigma v + \frac{1}{2}\rho(\vec{v} \times \vec{w})$

Radius $\frac{1}{2}|\rho|\,\|\vec{v} \times \vec{w}\|$

And we substitute $\vec{a} = <1, 4, 5>$, $\vec{b} = <-12, 8, 17>$, $\vec{v}$ $<1, 2, -3>$; $\vec{w} = <4, -1, 1>$

$\vec{v} \times \vec{w} = <-1, -13, -9>$ to obtain a system. $\begin{cases} 13 = \rho - \sigma + 4\tau \\ -4 = 13\rho - 2\sigma + \tau \\ -12 = 9\rho + 3\sigma + \tau \end{cases}$

which gives the solution.

$$\rho = \frac{-147}{251},\quad \sigma = -\frac{782}{251},\quad \tau = \frac{657}{251}$$

Center of the sphese: $\frac{1}{502} <-915, 791, 8525>$

$(\text{Radius})^2$: $\frac{1}{4}\rho^2 \|\vec{v} \times \vec{w}\|^2 = \left(\frac{147}{502}\right)(251) = \frac{147^2}{1004}$

$\therefore$ $(502x + 915)^2 + (502y - 791)^2 + (502z - 8525)^2 = 251(147)^2$.