CDT Lucas Enloe

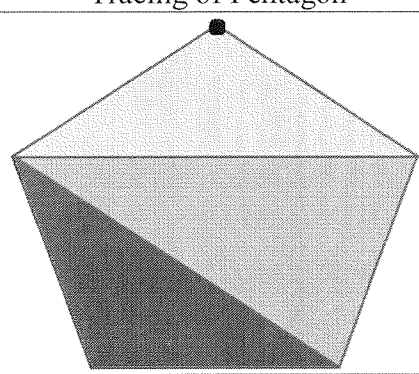
Problem 8: Double Counting

a) Ok, lets start with the pentagon. For every different subdivision of the pentagon into triangles, we'll follow a few rules:

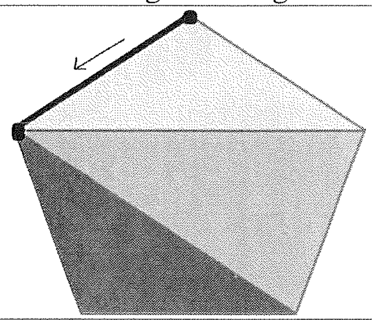
   a. Start at the top vertex, and trace the sides of the pentagon as you make your way counterclockwise to each vertex of the pentagon. For understanding, it helps if you actually use a pencil to trace around.
   b. Recognize that you are trying to form triangles, and that as you trace the pentagon, you are doing one of three things: opening a triangle, closing a triangle, or doing nothing.
   c. In order to open a triangle, you must trace its first side. In order to close a triangle, you must trace its second side. When you trace the triangle's second side, you automatically trace its third side (many times this will lead to you tracing through the pentagon, not around the perimeter).
   d. Every time you open a triangle, you insert an open parenthesis, and every time you close a triangle, you insert a closed parenthesis. By following these rules, every variation of the fragmented pentagon will give you every variation of 6 parentheses, correctly matched.

   Lets do an example, using the first subdivision in the problem statement:

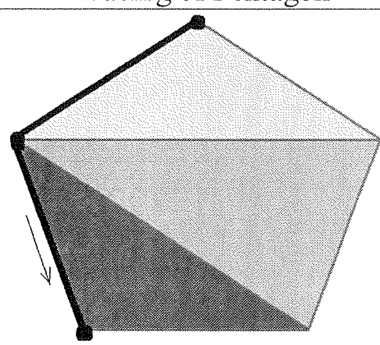
Step 1:

<table>
<tr><th>Tracing of Pentagon</th><th>Parentheses</th></tr>
<tr><td></td><td></td></tr>
<tr><td colspan="2">We start at the top vertex of the pentagon. We have not drawn any lines, so we don't have any parentheses yet.</td></tr>
</table>

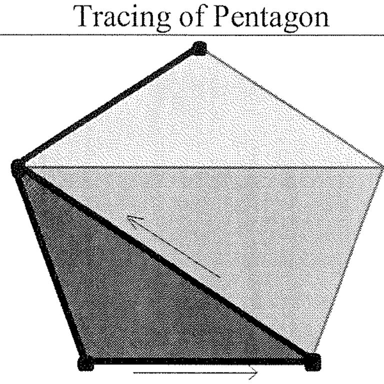
Step 2:

<table>
<tr><th>Tracing of Pentagon</th><th>Parentheses</th></tr>
<tr><td></td><td>(</td></tr>
<tr><td colspan="2">We trace to the next vertex of the pentagon. Notice that this has traced the first side of the light triangle. Because we have "opened" a triangle, we write an open parenthesis.</td></tr>
</table>

Step 3:

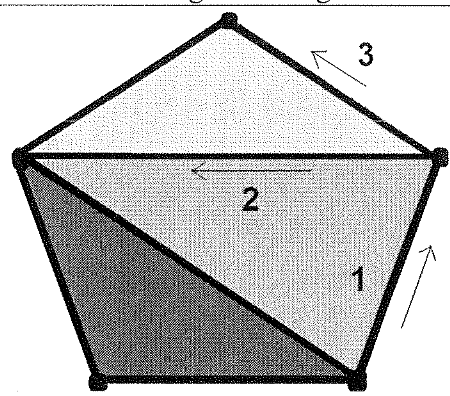
<table>
<tr><th>Tracing of Pentagon</th><th>Parentheses</th></tr>
<tr><td></td><td>((</td></tr>
<tr><td colspan="2">We again trace to the next vertex. Again, we have drawn the first side of a triangle, this time the dark one, so we add another open parenthesis.</td></tr>
</table>

Step 4:

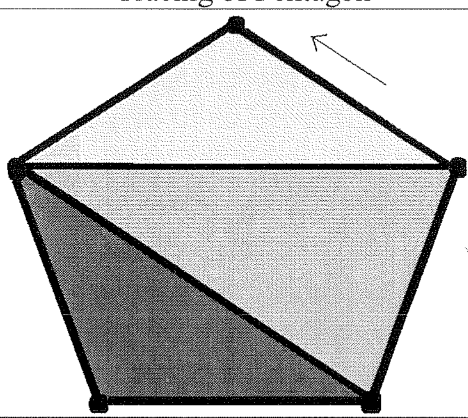
<table>
<tr><th>Tracing of Pentagon</th><th>Parentheses</th></tr>
<tr><td></td><td>(( ) (</td></tr>
<tr><td colspan="2">Ok, things get a little trickier now. We trace to the next vertex, which gives us the second side of the dark triangle. We can now "close" the triangle by drawing the third side of the dark triangle. Note that by drawing the third side of the dark triangle, we have drawn the</td></tr>
</table>

first side of the orange triangle. So, in this move, we have closed a triangle, and then opened one. This means that we write a closed parenthesis, then an open one.

Step 5:

<table>
<tr><th>Tracing of Pentagon</th><th>Parentheses</th></tr>
<tr><td>
</td><td>(( ) ( ))</td></tr>
<tr><td colspan="2">We continue along our trip around the pentagon, moving to the fourth vertex. By doing this, we draw the second side of the orange triangle. See how by drawing the third side of the orange triangle, we draw the second side of the light triangle, thus allowing us to close the light triangle as well. This leads to two consecutive closed parentheses.</td></tr>
</table>

Step 6:

<table>
<tr><th>Tracing of Pentagon</th><th>Parentheses</th></tr>
<tr><td></td><td>(( ) ( ))</td></tr>
<tr><td colspan="2">If you really want to be a stickler about it, you can trace the final side, but you didn't really do anything. Either way, by following our tracing rules, we've turned a fragmented pentagon into a set of matched parentheses. Useful? Not particularly. Cool? Most definitely.</td></tr>
</table>

By using this method for each separate variation of the pentagon, we get every variation of the parentheses. Sweet!

b) Well, I have no cool drawing method to answer this question. In fact, I have no method at all. Props[1], though, go to a Mr. David Pengelley and the Journal of Pure and Applied Mathematics, who *did* have a method. Even though I can't possibley understand the process, I can understand the final equation that the process led to:

$$P_{n+1} = \frac{(4n-6)P_n}{n}$$

Where P_n is the number of different ways to subdivide a polygon into triangles, and n is the number of sides of that polygon. With polygons of small side number, it is easy to find what P_5 is. Actually, with Excel, it isn't that much harder to find P_n for polygons with large side numbers too.

$P_{15} = 742900$

As a side note, I wanted to point out that the number of subdivisions grows *really* fast as number of sides increases:

Sides	Subdivisions
3	1
4	2
5	5
6	14
7	42
8	132
9	429
10	1430
11	4862
12	16796
13	58786
14	208012
15	742900
16	2674440
17	9694845
18	35357670
19	129644790
20	477638700
20	1767263190
20	6538873803
20	2.4194E+10
20	8.9517E+10
20	3.3121E+11

[1] This is how us young folk document our work these days. If only the Dean would get on board.