One thousand bored cadets are in the barracks, each in a room numbered between 1 and 1000. The first cadet (the most bored, and thus in room number 1) leaves his room and opens the door to every room. The second cadet, in room number two, goes out and closes the door to every room with an even number. The third cadet, in room number three, examines the doors of the rooms divisible by three, opening all of these that are open and closing all of them that are closed. The other cadets proceed, in order, in the same manner, changing the state of each door to a room which has a number divisible by their own room number. How many doors are open after the last cadet is finished?

Extra Credit: Which are which? In other words, describe an exact way to determine if a door is open or closed. For lots of extra credit, make it work in general, if there are n rooms, rather than 1000.

Solution:

- The short answer is 31 doors are open.
- The open doors are the doors whose numbers are perfect squares, i.e., $1, 4, 9, 16, \ldots, 961$.
- For any number of doors, simply take $\lfloor \sqrt{n} \rfloor$, where $\lfloor x \rfloor$ is the largest integer less than x.

We first need to determine if a door will be open or closed. Let's take room k. It will be opened or closed once for each number that divides it. Let $d(k)$ be the number of divisors of k. If $d(k)$ is odd, then the door will be open and if $d(k)$ is even, then it will be closed.

We can easily see that $d(k)$ is a multiplicative function, which means $d(k) = d(a) \cdot d(b)$ whenever $\gcd(a, b) = 1$. This is because we will get each divisor of a multiplied by all the divisors of b to give the divisors of k and since they are coprime, there can be no overlap of primes. So, we can determine this number by the prime decomposition of k.

So, consider the problem of $d(p^a)$. After a few examples, we see that $d(p^a) = a + 1$, since the divisors are $1, p, p^2, p^3, \ldots p^a$. Now, we just need to decide which values of k make $d(k)$ even. Let

$$k = p_1^{a_1} \cdot p_2^{a_2} \cdot \ldots \cdot p_m^{a_m},$$

where each p_i is distinct. We have that

$$d(k) = d(p_1^{a_1}) \cdot d(p_2^{a_2}) \ldots \cdot d(p_m^{a_m}) = (a_1 + 1) \cdot (a_2 + 1) \cdot \ldots \cdot (a_m + 1).$$

If any of the a_i's are odd, then we will have a closed door, because $d(k)$ will be even, so we must have all a_i's be even. Therefore, k can be written as a product of primes with even exponents,

$$k = p_1^{2b_1} \cdot p_2^{2b_2} \cdot \ldots \cdot p_m^{2b_m} = (p_1^{b_1} \cdot p_2^{b_2} \cdot \ldots \cdot p_m^{b_m})(p_1^{b_1} \cdot p_2^{b_2} \cdot \ldots \cdot p_m^{b_m}) = (p_1^{b_1} \cdot p_2^{b_2} \cdot \ldots \cdot p_m^{b_m})^2.$$

So, k is a square.

In order to answer the question, we need to know how many square numbers are less than n. Well, in our example $\sqrt{1000} = 31.6228$, so the squares of $1, 2, 3, \ldots, 31$ are all less than 1000. Therefore we find that for any n rooms, we will have $\lfloor \sqrt{n} \rfloor$ open rooms.