POTW #09-08 Shoes & Socks

Counting The Ways To Put On Shoes & Socks

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Problem

1. A cadet has two legs and needs to put on shoes and socks every morning. How many different orders are there for him to do this?
2. The cadet's crazy aunt has a dog which has four legs and she puts shoes and socks on each leg every day. How many orders are there in which she may do this? (The dog is very used to this and just puts up with it).
3. Watching the aunt is a spider with eight legs. The spider decides that if a dog can wear shoes and socks then so can a spider. How many ways are there for the spider to put on her shoes and socks? The spider has been watching and knows that she must put a sock on before a shoe.
4. The spider decides to show off her shoes and socks to a friend the caterpillar with n legs (the spider doesn't know how many legs the caterpillar has, but he has so many that it's possible one had to be amputated due to injury). The caterpillar decides to start wearing shoes and socks too! How many ways are there for him to put on the shoes and socks?

Solution

All four of these problems can be answered with a single line of reasoning. We assume that a sock must always be put on any foot before a shoe is put on that same foot.

Assume that a creature has n legs. We let sock_t and shoe_t represent the sock and shoe that go onto leg t. One ordering for these items would then be $\{\text{sock}_1, \text{shoe}_1, \text{sock}_2, \text{shoe}_2, \ldots, \text{sock}_n, \text{shoe}_n\}$. Notice that sock_t must precede shoe_t for the permutation to meet the requirement that a sock must be put on before a shoe. Now there are $(2n)!$ arrangements of these $2n$ items, but sock_1 precedes shoe_1 in only one-half of these cases. In a similar fashion sock_2 precedes shoe_2 in only one-half the cases, sock_3 precedes shoe_3 in one-half the cases, and so on. Therefore, we have $(2n)!/2^n$ countable orders that satisfy the requirement that socks must always go on before shoes.

Applying this reasoning to the above four problems we have the following results.

- **The Cadet**

Here $n = 2$ so we have 6 possible orders in which shoes and socks could be put on.

$$\frac{(2n)!}{2^n} \;/.\; n \to 2$$

6

- **The Dog**

Here $n = 4$ so we have 2520 possible orders in which shoes and socks could be put on.

$$\frac{(2\,n)!}{2^n} /. n \to 4$$

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2520
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The Spider

Here $n = 8$ so we have 81 729 648 000 possible orders in which shoes and socks could be put on.

$$\frac{(2\,n)!}{2^n} /. n \to 8$$

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81 729 648 000
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The Caterpillar

Here we have n legs so the number of possible orders in which shoes and socks could be put on then has the following general formula:

$$\frac{(2\,n)!}{2^n};$$