1.3 Properties of Linear Functions

We have discussed problem solving and the use of functions as tools to solve problems. We will now begin developing our skills in using models to solve problems. We typically begin the modeling process with simple models and build progressively better and more complex models. In fact, we may make several trips around the modeling triangle, building complexity and realism with each trip. Before we can build complexity, we must gain familiarity with the simplest of models, the linear model.

To illustrate the use of linear functions in modeling, we will use a story that is constantly in the news: the supply and demand that occurs as a result of the market price of oil per barrel. In this subsection we will look at the suppliers and consumers of a particular product whose production requires oil. Thus, when the price of oil rises the production cost also rises. We will focus on four related quantities:

- The **unit cost** of producing, transporting, and selling the product. As we build our model, we will use the letter c to denote this cost. We will analyze a situation in which the unit cost for the product in question is currently \$10.00 but that due to an anticipated rise in the price of oil the unit cost will rise to \$12.00.

- The **selling price** of the product. We will use the letter p to denote the selling price of the product in dollars.

- The **demand** for the product. This is the number of thousands of units of the product sold each week. It depends on the selling price. Consumers usually buy more when prices are low and buy less when prices are high. The relationship between the price and demand is called the *demand function.* We will denote this function *D(p).* Companies often do a great deal of market research to determine the demand function because it helps them to set their price and to determine their production. In this subsection we will assume that prior analysis has helped determine the demand function to be:

$$D(p) = \begin{cases} 1000 - 25p, & 0 \le p \le 40 \\ 0, & p > 40 \end{cases}. \quad (1)$$

Note that the price p is expressed in dollars and the demand *D(p)* is expressed in thousands of units per week. Figure 1.7 shows a graph of this function. Notice that as the price rises, as expected the demand goes down.

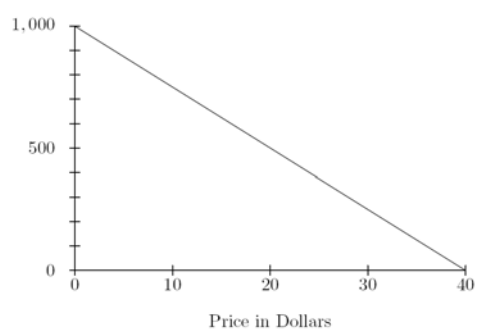

Figure 1.7: The Demand Function $D(p)$ (thousands of units per week)

- The **supply** of the product. This is the number (in thousands of units) of the product that are made each week. It is described by a function *S(p)* because the supply depends on the price. Producers usually produce more when the price is high and less when the price is low. Products are produced and sold in many different kinds of marketplaces. For some products there are a small number of large producers. For other products there are many smaller producers. Some products can be made easily and new producers can enter (or leave) the business easily. Other products are more difficult to make. For this example, we will assume that the product is easily made and is made by a large number of small producers who can enter or leave the market easily and who can easily increase or decrease their production. In this subsection we will assume that market analysis has helped determine the supply function to be a piecewise function, as defined in the previous section:

$$S(p) = \begin{cases} 0, & x < 10 \\ 40(p-10), & x \geq 10 \end{cases}. \qquad (2)$$

Figure 1.8 shows the supply function on the same set of axes as the demand function.

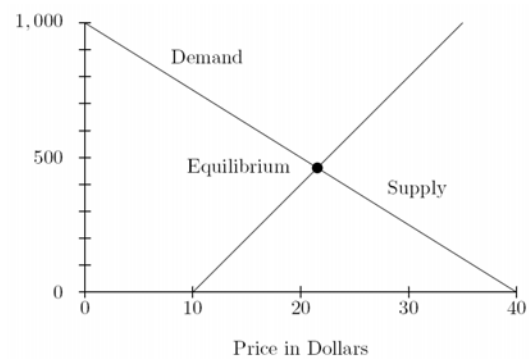

Figure 1.8: The Supply and Demand Functions (thousands of units per week)

Before going on, notice that we have just completed the first leg (transform) of the modeling triangle. We have been **given** our supply and demand functions, based on previous analysis. These functions provide us with more information about the system. In fact, \$10.00 in the supply function, $S(p) = 40(p-10)$, is the same number as the cost of producing, transporting, and selling our product. When the selling price is equal to this cost, the supply is zero because there is no point in producing a product if you can't make a profit. Notice also that as the price increases, the supply increases as well.

Now we are ready to do some mathematical analysis. Notice that when the price is low, the demand is above the supply, and when the price is high, the demand is below the supply. There is a point, marked by a dot in Figure 1.8 at which the supply and demand are equal. This point is called an equilibrium value because the producers will produce exactly enough units each week to fulfill the demand. Customers will be able to purchase exactly the number they desire. This is naturally the point we would like to **find.** We will discuss more about this application of the modeling triangle after we go into more detail about linear functions.

We say that the demand function depicted in Figure 1.7 and the supply function depicted in Figure 1.8 are "piecewise linear functions"; that is, the function is made up of more than one straight line, depending on the domain value.

Functions which have the same **average rate of change** on every interval are defined as *linear functions*.

The **average rate of change** of a function is the **rate of change** of a function, *f(x)*, over an interval, I, where $I = b - a$, in which b is the right limit of the domain and a is the left limit.

$$\text{Average Rate of Change} = \frac{\text{change in } f}{I} = \frac{\Delta \text{dependent variable}}{\Delta \text{independent variable}} = \frac{\Delta\, rise}{\Delta\, run} = \frac{f(b) - f(a)}{b - a}$$

The average rate of change of a function over an interval allows us to make a further classification of the function with which we're working. A function is said to be an **increasing function** if the average rate of change of *f(x)* is positive on every interval. Conversely, a function is classified as a **decreasing function** if the average rate of change of *f(x)* is negative on every interval. If the function is not increasing (or decreasing) on every interval, then it cannot be classified as such.

A *linear function* is a function that is in the form

$$f(x) = y = ax + d\,, \qquad (3)$$

where x is the independent variable and *f(x)* or y is the dependent variable; a is the average rate of change, or the **slope,** and d is the **y-intercept**. In previous courses, you may have used the equation $y = mx + b$ to describe a line. We will use equation (3) for the general form of the line for consistency with other functions throughout the course.

A variable represents an unknown quantity, or a quantity that *varies*. In equation (3) our x and y variables may take on many values because there is no restriction on the domain. However, for a given model the a and d quantities will always remain the same (constant). A fixed value within a function, like a or d in equation (3) is called a **parameter**.

Take some time right now to navigate to the interactive website located below to answer the question that follows.

http://www.dean.usma.edu/departments/math/MRCW/MA103/linear/live_graph.html

Question 1 *Select values for the slope of a line so that it is positive, negative, and zero. For which value is the function increasing? Decreasing? Neither?*

You should have noted from the interactive website that the larger the value of d is, the higher the intercept will be on the y-axis. The smaller the value of d is, the lower the intercept will be on the y-axis. We can conclude that the a and d parameters for the linear function determine its graph's slope (or shape) and location above (or below) the y-axis, respectively. In general, the parameters of any function (linear, trigonometric, exponential, etc.) determine its shape and location.

Parameters determine the *shape* and *location* of a function.

Since the parameter a in Equation 1 is constant, the graph of this function will always be a straight line. The sign of this parameter will determine if the line is increasing or decreasing. The magnitude of the parameter will determine how steep the line is. To demonstrate, let's consider examples.

Example 1 Suppose you have the three tables of information shown in Figure 1.9. Determine a function that describes the information in Table A. Plot and label the function.

Table A

x	$f(x)$
-5	-7.5
-4	-6
-3	-4.5
-2	-3
-1	-1.5
0	0
1	1.5
2	3
3	4.5
4	6
5	7.5

Table B

h	$g(h)$
-5	-2.5
-4	-2
-3	-1.5
-2	-1
-1	-0.5
0	0
1	0.5
2	1
3	1.5
4	2
5	2.5

Table C

t	$s(t)$
-5	10
-4	8
-3	6
-2	4
-1	2
0	0
1	-2
2	-4
3	-6
4	-8
5	-10

Figure 1.9: Data for Example 1

Step 1: *Transform the problem.* Let's consider Table A. We are **given** a table of data; the first step is to plot the data so we can see if it exhibits a pattern.

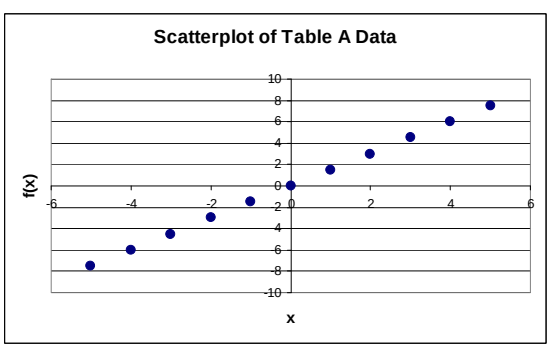

Figure 1.10: Scatterplot of Table A Data.

The data in Table A certainly exhibits a linear pattern, so we will continue the process of **finding** a line that fits through the data. We see that our data contains the point (0, 0), the origin. Because the data goes through the origin, the *y*-intercept, or 'd' value in the linear equation, will be equal to zero. Since the *y*-intercept is zero, there is one parameter left to find: our **plan** is to find the average rate of change, the slope of the line.

Step 2. *Solve the problem using appropriate solution techniques.* To **solve**, calculate the rate of change of the data at each point, using the formula on page 18. $Slope = \frac{\Delta\, rise}{\Delta\, run} = \frac{\Delta\, f(x)}{\Delta\, x}$. So, we see that the slope between the first two data points is $\frac{-6-(-7.5)}{-4-(-5)} = \frac{1.5}{1} = 1.5$. The slope between the remaining points is calculated in the same manner and works out to be the same value, 1.5.

Step 3. *Interpret the Solution.* The linear function we have developed to model the data in Table A is $f(x) = 1.5x + 0,\ or\ f(x) = 1.5x$. To **reflect** on the solution we have attained, it will be most helpful to draw a picture to make sure that our linear function goes through our data points. See Figure 1.11.

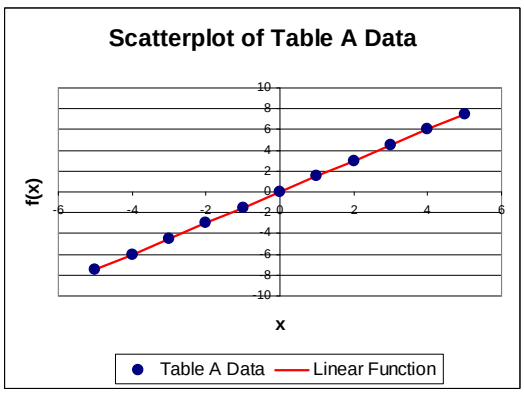

Figure 1.11: Linear Model Plotted Against Table A Data

The linear function developed to model the data we were given seems to fit the data points very well; therefore, we are satisfied that $f(x) = 1.5x$ is indeed a reasonable function to model this data set. The definition of a function is incomplete with knowing the domain and range that it is good for. In the case of this specific function, the domain and range are as follows:

$$Domain: \{x \mid -5 \le x \le 5\}$$
$$Range: \{y \mid -7.5 \le y \le 7.5, where\ y = f(x)$$

Question 2 *What are the functions that would best model the data sets given in Tables B and C in Figure 1.9?*

Now that we understand the properties of linear functions, let's use linear functions we have seen already, the supply and demand relationships established in equations (1) and (2), along with the modeling triangle and problem solving process described in Section 1.1, to solve a problem of interest.

Example 2 A supplier of goods is interested in the maximum price possible to sell a commodity at which there will be no wasted product. The price in a supply-demand system at which the supplier sells all possible product and consumers are able to buy as much as they want is the equilibrium price. What is the equilibrium point of the system described at the beginning of section 1.3?

REAL WORLD PROBLEM: Find an equilibrium point for a given supply and demand system.

Step 1. Transform problem into a mathematical model.

a. **Given:** Relationship of demand to price: $D(p) = 1000 - 25p$

Relationship of supply to selling price: $S(p) = \begin{cases} 0, & x < 10 \\ 40(p-10), & x \geq 10 \end{cases}$

Variable declaration: *D(p)* = Demand (thousands of units) as a function of selling price.

S(p) = Supply (thousands of units) as a function of selling price

Definition of equilibrium price: *D(p) = S(p)*

b. **Find:** Equilibrium price as defined above.

c. Assume: Selling price is greater than $10, so supply is greater than zero.

d. Solution **Plan**: Use algebraic manipulation to solve supply and demand

equations: $D(p) = 1000 - 25p = S(p) = 40(p-10)$
$1000 - 25p = 40p - 400$

Step 2. **Solve** using appropriate solution techniques (algebraic manipulation).

$$1000 - 25p = 40p - 400$$
$$1400 = 65p$$
$$p = 21.54$$

MATHEMATICAL SOLUTION: Equilibrium price = $21.54

Step 3. Communicate and **reflect** upon results.

The equilibrium selling price of $21.54 is a reasonable price; it seems to match closely with the intersection of the supply and demand equations plotted in Figure 1.8.

In this lesson, we have focused on the *slope-intercept* form of a line. There are two other forms, the *point-slope* form and the *general form* of a line.

> The *slope-intercept form* is: $f(x) = ax + d$, where a is the slope (rate of change) and d is the y-intercept.
>
> The *point-slope* form of a line is: $y - y_0 = a(x - x_0)$, where a is the slope (rate of change) and (x_0, y_0) is a point on the line.
>
> The *general form* of a line is : $Ax+By+C=0$, where A, B, and C are constants.

Question 3 *Using the definition for average rate of change, your understanding of y-intercept, and your knowledge of the slope-intercept form of a line,*

a. Determine an equation describing each of the following tables of data.

Table A

x	$f(x)$
-5	-2
-4	-1.6
-3	-1.2
-2	-0.8
-1	-0.4
0	0
1	0.4
2	0.8
3	1.2
4	1.6
5	2

Table B

h	$g(h)$
-5	5.5
-4	5
-3	4.5
-2	4
-1	3.5
0	3
1	2.5
2	2
3	1.5
4	1
5	0.5

Table C

t	$s(t)$
-5	-13
-4	-11.4
-3	-9.8
-2	-8.2
-1	-6.6
0	-5
1	-3.4
2	-1.8
3	-0.2
4	1.4
5	3

b. Verify your equations for each table of data. Then, graph and label each function.

***Question 4** Thayer Hall has four floors and is approximately 70 feet tall. Taylor Hall has nine floors and is approximately 180 feet tall. Develop a model that predicts the height of a building based on the number of floors it has. (Don't forget to include your model's domain and range.)*

Question 5 *Plot two lines that are parallel. What is relationship of their slopes? Ensure you draw the lines to scale as much as possible (graph paper helps).*

Question 6 *Plot two lines that are perpendicular. What is relationship of their slopes? Ensure you draw the lines to scale as much as possible (graph paper helps).*