1.6 The Generalized Exponential Function

Take a moment to conduct an internet search on the word "exponential." When this chapter of the text was being written, Google returned over 7 million hits on sites or articles that somehow referenced "exponential." Below are just a few of the quotes you may find.

"The death toll from the Indian Ocean tsunami is set to rise ***exponentially*** *above current estimates of 150,000 as relief workers reach remote villages and survivors succumb to disease, UN officials warned today."*[1]

"Scientists are finding that electromagnetic flux of unknown spectrum in action may be the root cause of the ***exponentially*** *increasing number of earthquakes in the last eighteen months."*[2]

Exponential growth or decay…you hear the term used by newscasters, political pundits, and people you talk with, but what precisely does it mean?

You may have personal experience with exponential functions in a place that may be important to you in the future - an interest bearing savings account. Perhaps you have taken a class at some point in your life where you learned a function that allowed you to compute the value of an account that accrued interest continually. You may have heard it referred to as PERT, or seen it presented as Equation (1); this equation is an exponential function.

$$V = Pe^{rt} \qquad (1)$$

where:

V is the value of the account at the time that you are interested (dependent variable)

P is the principal, or initial value invested (parameter)

e is the **base** (parameter)

r is the interest rate (parameter)

t is the time that has passed (in the same units as the period referred to in *rate*) since the initial investment. (independent variable)

What makes this function exponential is that there is a constant base raised to a variable power. In this case, the base is e which is also approximately 2.71828. The most basic form of the exponential function is:

$$f(x) = b^x \qquad (2)$$

where b is the constant base (parameter) raised to a variable power x, the independent variable. **Do not confuse this with the power function (learned later) where the independent variable is the base. Remember: the exponential function has the independent variable in the exponent.**

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[1] http://www.guardian.co.uk/tsunami/story/0,15671,1382973,00.html

[2] qd.typepad.com/19/2005/01/its_been_a_whol.html

1.6.1 Properties of Functions

Recall that a function is said to be an increasing function if the average rate of change of *f(x)* is positive on every interval. Conversely, a function is classified as a decreasing function if the average rate of change of *f(x)* is negative on every interval. Similarly, we say a function is increasing on an interval if the average rate of change of *f(x)* is positive on the interval. Also, a function is decreasing on an interval if the average rate of change of *f(x)* is negative on every interval. More precisely,

A function is **increasing** on an interval if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ and x_1, x_2 are in the interval.
A function is **decreasing** on an interval if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$ and x_1, x_2 are in the interval.

If we trace a graph from left to right, we notice our hand will move up when the function is increasing and down when decreasing. In Section 1.3, we saw that a linear function increases when it has a positive slope and decreases with a negative slope. This extends to other functions – a positive rate of change indicates the function is increasing and negative, decreasing.

In the linear case, we saw that the rate of change was constant and its sign determined if the line was increasing or decreasing. We will see many examples of exponential functions that increase or decrease and their rates of change that also increase or decrease. The next subsection catalogs the three cases of exponential functions and their corresponding changing rates.

1.6.2 Three Cases of the Basic Exponential Function, b^x

There are three cases of this basic exponential function; each case has a different value for *b*.

Case 1: $b > 1$. This function increases as the domain values increase. This can be seen in the first table of Figure 1.24a. This case of the exponential function also increases at an increasing rate. This can be seen by investigating the average rates of change which are also shown in Figure 1.24a. Notice how these rates are increasing. Thus the function is not only increasing but increasing at an increasing rate. A graph of this type of function is shown in Figure 1.24b. These functions are normally referred to as growth functions.

Case 2: $0 > b > 1$. This function decreases as the domain values increase. This can be seen in the second table of Figure 1.24a. This case of the exponential function also decreases at an increasing rate. This can be seen by investigating the average rates of change. Notice in the second table of Figure 1.24a how these rates are increasing. Thus the function is not only decreasing but decreasing at an increasing rate. These functions are normally referred to as decay functions.

Case 3: $b = 1$. This function equals 1 everywhere. The average rate of change of this function will always be zero (Figure 1.24a) regardless of whether or not the domain values increase or decrease. Since this case results in a constant function (Figure 1.24), we will not investigate this case any further in this section.

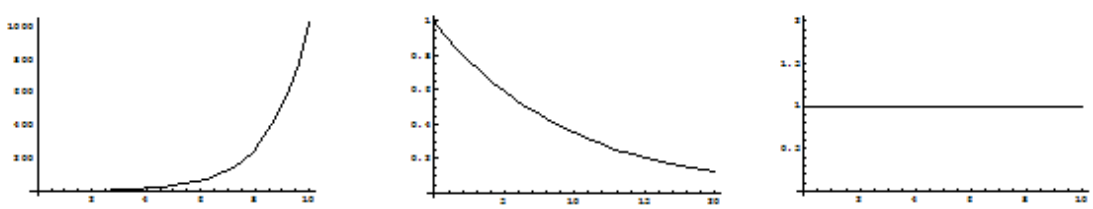
NOTE We only consider values for b that are greater than or equal to zero. Why? Well, what happens when you raise a negative real number to a power? If the exponent is an even integer, then the answer is a positive real number. If the exponent is an odd integer, then the answer is a negative real number. If the exponent is any non-integer real number, then the answer is a complex number. In this course, we will focus on real-valued functions. That is, we will concentrate on functions whose domains and ranges are both subsets of the real numbers.

$b = 2$		
x	$f(x)=b^x$	*AVG RoC*
-2	0.250	
-1.5	0.354	0.207
-1	0.500	0.293
-0.5	0.707	0.414
0	1.000	0.586
0.5	1.414	0.828
1	2.000	1.172
1.5	2.828	1.657
2	4.000	2.343

$b = .5$		
x	$f(x)=b^x$	*AVG RoC*
-2	4.000	
-1.5	2.828	-2.343
-1	2.000	-1.657
-0.5	1.414	-1.172
0	1.000	-0.828
0.5	0.707	-0.586
1	0.500	-0.414
1.5	0.354	-0.293
2	0.250	-0.207

$b = 1$		
x	$f(x)=b^x$	*AVG RoC*
-2	1	
-1.5	1	0
-1	1	0
-0.5	1	0
0	1	0
0.5	1	0
1	1	0
1.5	1	0
2	1	0

Figure 1.24a: Average Rates of Change for Exponential Functions (Cases 1-3) on Domain -2 < x < 2



Case 1: Increasing at an Increasing Rate

Case 2: Decreasing at an Increasing Rate

Case 3: Constant Function (Rate of Change = 0)

Figure 1.24b: Graphs Depicting Basic Exponential Functions

Question 1 *Plot graphs of each of the three cases for exponential functions, different from those shown in Figure 1.24b. Evaluate the functions over the domain [-3, 3] for each case. For Case 1, assume b = 1.5, for Case 2, assume b = .4, and for Case 3, assume b = 1. Identify the domain and range of each of these functions.*

1.6.3 The Generalized Exponential Function

The graphical and associated word descriptions of the behavior of exponential functions provided above will prove very useful in your modeling efforts. However, it is important to recognize other forms of the exponential function and their patterns of behavior as well. In this course, we will investigate the more generalized exponential function listed below.

$$f(x) = ab^x + d \qquad (3)$$

The three cases of the basic exponential function shown in Figure 1.24 share the property that the parameter $a = 1$ and that the parameter $d = 0$. The parameters for a generalized exponential function (in a similar manner to that for the linear function) control the shape and location of its graph.

Generalization 1. We can shift any graph of a function upward or downward by adding a parameter to the function:

$$y = f(x) + d$$

If d is positive, we will shift or translate the function's graph upward. If d is negative, we will shift or translate the function's graph downward. Unique to exponential functions is that the parameter d also indicates the location of a horizontal asymptote.

A **horizontal asymptote** of a curve is a line $y = L$ that the function *f(x)* approaches as x approaches infinity.
A **vertical asymptote** of a curve is a line $x = a$ where the values of the function *f(x)* approaches infinity as x approaches the undefined point a.

Generalization 2. By changing the a parameter, we can flip the function about the horizontal asymptote, or line $y = d$. We can also stretch and shrink the function vertically. Table 1.8 shows the vertical stretch and shrink of an exponential function, $f(x) = e^x$.

x	$f(x)=e^x$ "Base" Function	$f(x)=10e^x$ Vertical *Stretch*	$f(x)=0.10e^x$ Vertical *Shrink*
-2	.1353	1.353	.01353
-1	.3679	3.679	.03679
0	1	10	.1
1	2.7182	27.182	.27182
2	7.389	73.89	.7389

Table 1.8: Tabular Representation of Vertical Stretching and Shrinking

Note that the first column is the base function, the second includes a vertical *stretch*. The a parameter is increased, so for each value of the independent variable, the value of the vertically stretched function is larger than that of the base function. In the third column, we implement a vertical *shrink* by decreasing the value of the a parameter. For each value of the independent variable, the function value is smaller.

We can see the effect of changes made on the a parameter graphically, as well as in a table. The upper left picture in Figure 1.25 is $f(x)=e^x$. The upper right function shows the result of negating the function. In addition, we can vertically stretch or shrink the graph of any function by increasing or decreasing the a parameter. The bottom left function illustrates the effect of decreasing the value of the a parameter; the bottom right shows the effect of increasing the a parameter.

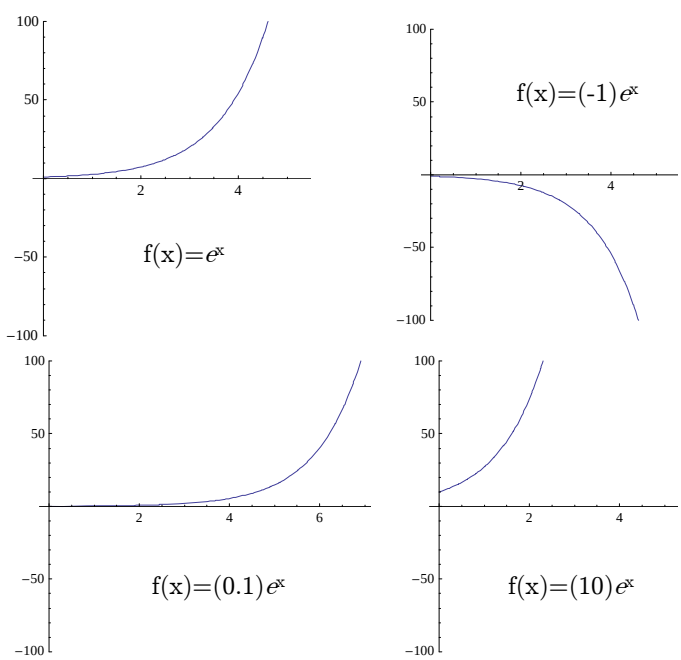


Figure 1.25: Illustrating the Effect of Changing the a Parameter in the Exponential Function

Generalization 3. The parameter b determines the shape of the function. Recall the three cases of the exponential function illustrated in Figure 1.24a and b. Another illustration of the impact of the b parameter is shown in Figure 1.26.

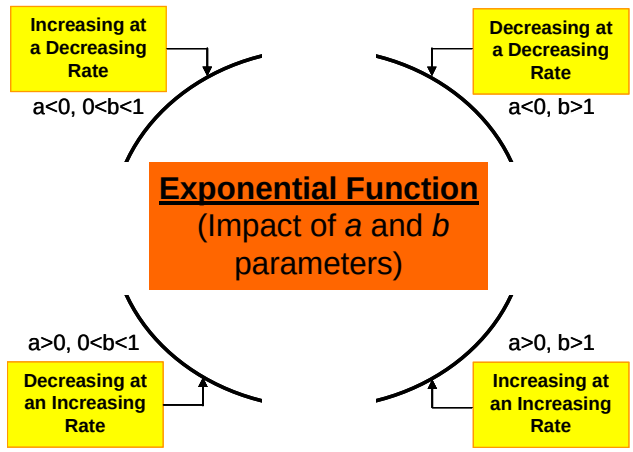


Figure 1.26: An Illustration of Generalizations 2 & 3.

The first case, increasing at a decreasing rate, means that our finger will move up as we trace left to right on the upper left curve in Figure 1.26 (increasing), but not as fast at the end as at the beginning. Next, decreasing at a decreasing rate means when traced, our finger moves down, but we actually decrease faster at the end. Let's explain in terms of the average rate of change, a concept we discussed in the context of linear models.

We see that the function in the upper right quadrant is decreasing; that is, the dependent variable is getting smaller as the independent variable increases. The next area to address is the rate at which the function is decreasing, an analysis of the rate of change. In doing this, always move from left to right on the path of the function.

First consider the ruler labeled "Step 1" in Figure 1.27, measuring the average rate of change between the points $(1, 7.5)$ and $(4, 7.0)$.

$$average\ rate\ of\ change = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7.0 - 7.5}{4 - 1} = \frac{.5}{3} = -.1\overline{66}$$

Next, consider the ruler labeled "Step 2" in Figure 1.27, measuring the average rate of change between the points (4, 7.0) and (7, 4.5).

$$average\ rate\ of\ change = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4.5 - 7.0}{7 - 4} = \frac{-2.5}{3} = -.8\overline{33}$$

Notice that -.833 < -.166, therefore the average rate of change (or the slope) is *decreasing* and the function is decreasing at a decreasing rate.

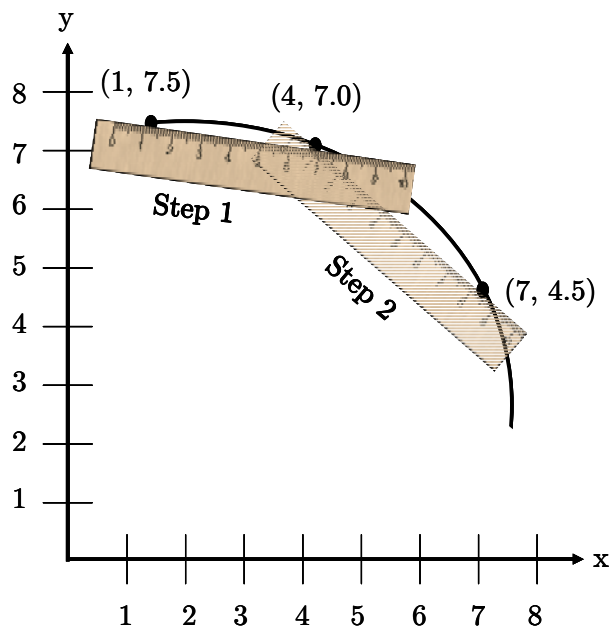


Figure 1.27: Illustration of Decreasing at a Decreasing Rate.

We will use Figures 1.26 and 1.27 to describe many standard functions in the upcoming lessons, including exponential functions in this section.

The box below summarizes the exponential properties we have discussed so far. You should have a good understanding of each of these.

Properties of Exponential Functions of the form $f(x) = ab^x + d$:

- If $a > 1$, and
  - $b > 1$, the function will increase at an increasing rate as the independent variable increases.
  - $0 < b < 1$, the function will decrease at an increasing rate as the independent variable increases.
  - $b = 1$, the function will remain constant an equal to the sum of the parameters a and d as the independent variable increases.
- If $a < 1$, and
  - $b > 1$, the function will decrease at a decreasing rate as the independent variable increases.
  - $0 < b < 1$, the function will increase at a decreasing rate as the independent variable increases.
  - $b = 1$, the function will remain constant an equal to the sum of the parameters a and d as the independent variable increases.

You should also have a good understanding of the Law of Exponents which is summarized in the following box.

Law of Exponents for Exponential Functions:

1) $a^{x+y} = a^x a^y$

2) $a^{x-y} = \dfrac{a^x}{a^y}$

3) $(a^x)^y = a^{xy}$

4) $(ab)^x = a^x b^x$

You may want to take some time right now to navigate to the interactive website located at:

http://www.dean.usma.edu/departments/math/MRCW/MA103/exponential/live graph.html

(You can also link to this website through the MA-103 course webpage). Once there, adjust the parameters to see how the graph changes. Intuitively, we see that if the equation had the values a=1, d=0, then we are back to our simplest type of exponential function described in Equation (2).

Question 2 *In your own words, describe how changes in the parameters of the generalized exponential function affect its shape.*

Question 3 *In Table 1.9 there are three sets of data that represent three different functions. Which data comes from an exponential function? Can you identify the other types of functions from the data? Do you need more information for the first two data sets?*

x	y
0	20.0000
1	21.0000
2	22.1000
3	23.2775
4	24.6425
5	26.2650

x	y
0	20.0000
1	21.0000
2	22.0500
3	23.1525
4	24.3101
5	25.5256

x	y
0	20.0000
1	21.0000
2	22.0000
3	23.0000
4	24.0000
5	25.0000

Table 1.9: Values for Question 3[3]

[3] This problem is from Functioning in the Real World, A Precalculus Experience by Gordon, Gordon, Tucker, and Siegel, pg. 83

1.6.4 Applications of the Generalized Exponential Function

Example 1 Modeling the Growth of an Investment.

Let's discuss the growth of money in a traditional savings account. If money in the account grew in a linear manner, the changes in the account from year to year, as we learned in Lesson 4, would all be the same (there would be a constant average rate of change). The second column of Table 1.11 illustrates how a $1000 investment might actually grow in a savings account. You can see in the third column, the average rate of change between each successive year is NOT constant and thus, the growth is NOT linear. In fact, the average rates of change are increasing because the more money that is in the account, the more money there is to make additional interest. The balance of the account can be modeled with a Case 1 exponential function where parameter $a > 0$ and parameter $b > 1$.

Suppose your parents put money into a bank account for you after you graduated the sixth grade. You are not sure what the interest rate is, but based upon old annual statements, you can see the growth that has occurred. You would like to know how much will be in that account when you retire from the military after 20 years of service (23 years from now, year 7).

Year	Amount	AVG RoC
0	$1,000.00	
1	$1,051.27	51.27
2	$1,105.17	53.90
3	$1,161.83	56.66
4	$1,221.40	59.57
5	$1,284.03	62.62
6	$1,349.86	65.83
7	$1,419.07	69.21

Table 1.10: Average Rate of Change of an Exponential Function

Step 1: *Transform the problem.* We are **given** a table of data, we must examine the table.

- Define the variables:
  - Independent variable (input): year
  - Dependent variable (output): amount of money in account

- Nature of the data: Apply the quantitative measures of the nature of the data to gain an idea of which type of function may be most appropriate. See Table 1.10.

Notice that the data are increasing, as is the rate; therefore, we need a function that is increasing at an increasing rate: a Case 1 exponential function. Let's graph the data to verify our conjecture. See Figure 1.28.

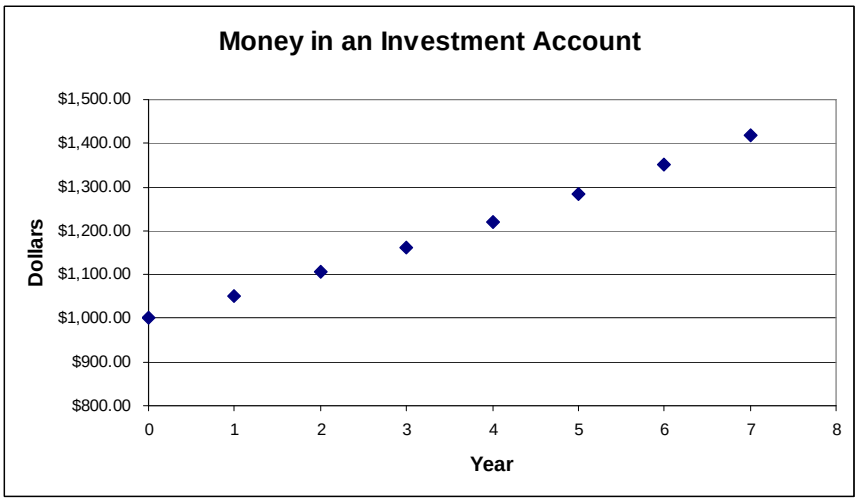


Figure 1.28: Scatterplot of Table 1.10 Data.

The data in Figure 1.28 seem to exhibit a *linear* pattern. Our quantitative measure of the nature of the data and our visual measure seem to conflict. We will explore the two models that explain what appears to be going on in the data set: the linear model (visual inspection) and exponential model (nature of the data). We know that we will have to **find** the model that fits through the data best.

- Assumptions
  - No money is withdrawn from the account
  - No additional deposits are made into the account
  - The data accurately represent what will happen to the population in the coming years.

Our plan is to use the model development and evaluation techniques that we have used to this point in the course to find the best model to determine how much money we will have after any given year in the future.

Step 2. *Solve the problem using appropriate solution techniques.* To **solve**, we will develop a linear model and an exponential model, then compare the two to decide which is best.

- Linear model development. First, we'll try the linear model. In general, our linear model will be:

$$Amount(year) = a(year) + d$$

<u>Linear Model Parameter Estimation</u>: Using modeling skills developed to date, we need to estimate the parameters (a and d) for a linear model. Let's use the first and last data point to form two equations we can solve for the two unknown parameters. The two equations follow:

$$1000 = a(0) + d$$
$$1419.07 = a(7) + d$$

We see by inspection that the d parameter (y-intercept) is 1000. We can use that information to solve for a.

$$1419.07 = a(7) + 1000$$
$$419.07 = 7a$$
$$a = \frac{419.07}{7} \approx 59.87$$

The estimated model is:

$$Amount(year) = 59.87(year) + 1000$$
$$Domain: \{year \mid 0 \le year \le 7\}$$
$$Range: \{Amount \mid 1000 \le Amount \le 1419.07$$

Overlaid with our data, the model is graphed in Figure 1.29.

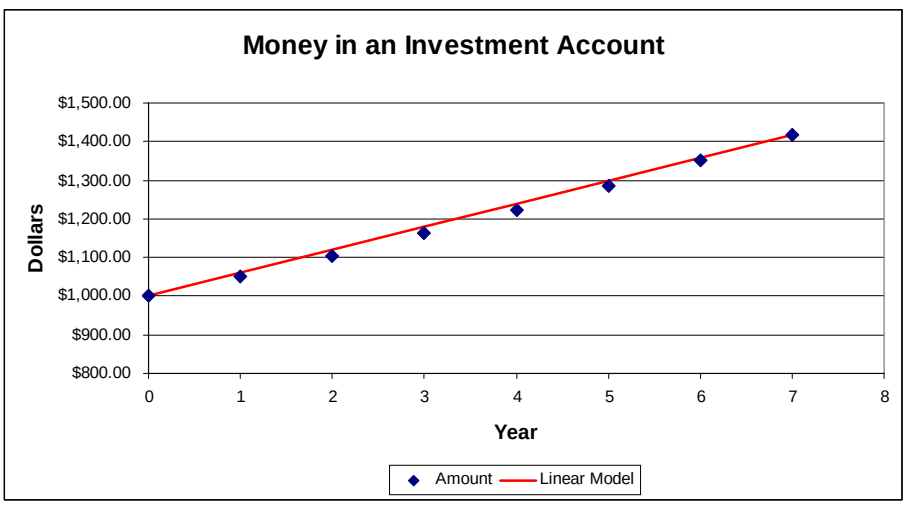


Figure 1.29: Investment Data Overlaid with Linear Model.

Using the eyeball test to evaluate Figure 1.29, we see that the estimated linear model seems to be a pretty good fit to the data.

Question 4 *Calculate the Sum of Squared Error and the coefficient of determination for the linear model shown in Figure 1.29. Use the two tests to determine the model's "goodness of fit."*

- Exponential model development. Now that we've developed a power model, it is time to develop our generalized exponential model, of the form:

$$Amount(year) = ab^x + d$$

As we saw earlier in the example, each of the points seems representative. It make sense to solve for all three parameters if we can, so we'll use the first, last, and middle points to form three equations in three unknowns and let Mathematica do the work for us. The three equations are:

$$1051.27 = ab^1 + d$$
$$1221.40 = ab^4 + d$$
$$1419.07 = ab^7 + d$$

```
In[1]:= Solve[{1051.27 == a * b^1 + d, 1221.40 == a * b^4 + d, 1419.07 == a * b^7 + d},
        {a, b, d}]

Out[1]= {{d → 0.281732, a → -499.859 - 865.782 i, b → -0.525642 + 0.910438 i},
        {d → 0.281732, a → -499.859 + 865.782 i, b → -0.525642 - 0.910438 i},
        {d → 0.281732, a → 999.719, b → 1.05128}}
```

Figure 1.30: Using Mathematica to Solve for Three Parameters

Using the solution to the systems of equations in Figure 1.30, yields the final model:

$$Amount(year) = 999.719(1.05128)^x + 0.281732$$
$$Domain: \{year \mid 0 \le year \le 7\}$$
$$Range: \{Amount \mid 1000 \le Amount \le 1419.07$$

What if we encounter an example (there are many) where Mathematica cannot solve for three parameters simultaneously?

Let's assume a value for the b parameter to make the function *intrinsically linear* (in the form $ax+d$) and then solve for a and d. For this example, we see that the rate at which the function is increasing is not significant, so we could assume a value for b that is close to 1. Let's assume $b = 1.01$.

> A function is said to be ***intrinsically linear*** when it can be written in the form $f(x) = ax+d$. (Notice the form of the exponential function, after a 'b' parameter is estimated and values substituted in for representative points)
>
> $$1051.27 = a(1.01)^1 + d \rightarrow 1051.27 = a(1.01) + d \rightarrow f(x) = ax + d$$

Use the second and last data points to form two equations to solve for the two unknown parameters:

$$1051.27 = a(1.01)^1 + d$$
$$1419.07 = a(1.01)^7 + d$$

```
Solve[{1051.27 == a (1.01)^1 + d,  1419.07 == a (1.01)^7 + d}, {a, d}]

{{a → 5919.34, d → -4927.26}}
```

Figure 1.31: Solving for Two Parameters in Mathematica

Using the solution to the systems of equations in Figure 1.31, yields the final model:

$$Amount(year) = 5919.34(1.01)^x - 4927.26$$
$$Domain: \{year \mid 0 \le year \le 7\}$$
$$Range: \{Amount \mid 1000 \le Amount \le 1419.07$$

Let's plot our exponential models, Figure 1.32, to see how well it appears to fit the data and how well it compares to the linear model we already plotted.

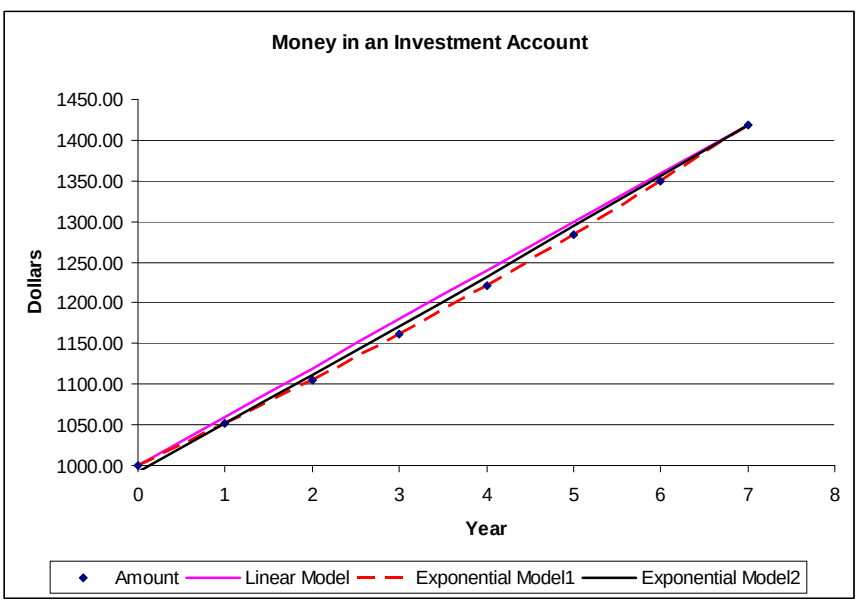


Figure 1.32: Investment Data Overlaid with Linear and Exponential Models

Step 3. *Interpret the results of the solution.* All models appear to fit the data relatively well, but which is best? Using the eyeball test, the exponential model1 seems to fit the data better than the estimated exponential model2 or the linear model. Because of the scale of the graph, we cannot be precisely sure that the eyeball test, a qualitative (subjective) measure is accurate

enough in this case. We will verify our conjecture using each model's sum of squared error and coefficient of determination. See the results in Figure 1.33.

Year	Amount	Linear Model	Mean Error	Squared Mean Error	Model Error	Squared Model Error		
0	1000.00	1000.00	-209.55	43909.11	0.00	0.00		Linear
1	1051.27	1059.87	-158.28	25050.98	-8.60	73.96	a	59.87
2	1105.17	1119.74	-104.38	10894.14	-14.57	212.28	b	
3	1161.83	1179.61	-47.71	2276.72	-17.78	316.13	c	
4	1221.40	1239.48	11.86	140.54	-18.08	326.89	d	1000
5	1284.03	1299.35	74.49	5548.02	-15.32	234.70		
6	1349.96	1359.22	140.42	19716.37	-9.26	85.75		
7	1419.07	1419.09	209.53	43900.73	-0.02	0.00		
	Mean:	1209.55	**SST:**	151436.60	**SSE:**	1249.71		
					r^2:	0.99174763		
		Exponential Model1						
		1000.00	-199.06	39625.61	0.00	0.00		Expo1
		1051.27	-147.79	21842.43	0.00	0.00	a	999.719
		1105.16	-93.89	8815.68	0.01	0.00	b	1.05128
		1161.82	-37.23	1386.21	0.01	0.00	c	
		1221.38	22.34	498.99	0.02	0.00	d	0.281732
		1284.00	84.97	7219.59	0.03	0.00		
		1349.83	150.90	22770.26	0.13	0.02		
		1419.03	220.01	48403.59	0.04	0.00		
	Mean:	1199.06	**SST:**	150562.36	**SSE:**	0.02		
					r^2:	0.99999987		
		Exponential Model2						
		992.08	-203.45	41392.98	7.92	62.73		Expo2
		1051.27	-152.18	23159.56	0.00	0.00	a	5919.34
		1111.06	-98.28	9659.48	-5.89	34.68	b	1.01
		1171.44	-41.62	1732.44	-9.61	92.39	c	
		1232.43	17.95	322.11	-11.03	121.64	d	-4927.26
		1294.03	80.58	6492.71	-10.00	99.92		
		1356.24	146.51	21464.40	-6.28	39.42		
		1419.07	215.62	46490.84	0.00	0.00		
	Mean:	1203.45	**SST:**	150714.53	**SSE:**	450.77		
					r^2:	0.99700912		

Figure 1.33: Computation of Linear and Exponential SSE and r^2

As expected, the exponential models have a lower SSE and a higher r^2, and are therefore, a better fit to the data than the linear model. Exponential model1 is better than the exponential model2 which is also to be expected since we estimated the parameter b in the set-up of exponential model2.

Question 5 *Complete the* ***reflect*** *step of the modeling process for Example 1. Does the model you chose to have the best fit make sense based upon the number of parameters and your knowledge of the nature of investment data?*

Question 6 *Complete a sensitivity analysis of the two estimated models. Do your findings change significantly?*

Example 2 Modeling Prozac in the Bloodstream

Let's now analyze decay functions by considering one of the most widely taken drugs to treat depression, Prozac. If a person takes a certain single dose, it will eventually be eliminated from the bloodstream by the kidneys. We can assume that during a given fixed time period, the kidneys will remove a certain percentage of the drug from the bloodstream. In fact, it has been found that the kidneys remove one-fourth of the drug during any 24-hour period so that 75% of the drug will still remain.

It is unhealthy for two different antidepressants to work in the body at the same time. In fact, a person can have no more than 10mg of Prozac in their blood to safely begin another drug regimen. Let's assume that a person must change the prescription they are on from Prozac to another drug. The pharmacist tested the patient to determine how much Prozac is currently in the blood. The test revealed an amount of 60mg. Given this initial amount of Prozac in the blood, it is your job to advise the pharmacist when to safely prescribe the new drug.

Step 1: *Transform the problem.* We are **given** the rate at which Prozac is removed from the bloodstream (75% eliminated per day). Therefore, we know that our b parameter is $b = .75$...a decaying exponential function. We are also given the start point; at day zero, there is 60mg of Prozac in the blood.

- Define the variables:
  - Independent variable (input): time (days)
  - Dependent variable (output): amount of Prozac in the blood

- Assume: For this example, we will assume that the parameter $d = 0$ because after an infinite time period, the level of the drug in the blood will tend toward zero (i.e., there is a horizontal asymptote at $y = 0$ – recall earlier we mentioned that the d parameter equates to the horizontal asymptote). Therefore, our new exponential function becomes:

$$druglevel(t) = ab^t + 0 \qquad (4)$$

The goal of this problem is to **find** how many days it will take the Prozac to reach a safe level to administer the new drug, a level of 10mg. Our **plan** will be to develop a model for Prozac being eliminated from the blood stream. We can then iterate the function to see when the function drops below 10mg of Prozac in the bloodstream.

Step 2. Solve the problem using appropriate techniques. Given that the initial dose is 60mg, and our horizontal asymptote is at $t=0$, we are able to solve for the a parameter as follows:

$$druglevel(t) = ab^t$$
$$druglevel(0) = 60 = ab^0$$
$$60 = a(1)$$
$$60 = a.$$

And in general, the following function models the amount of Prozac in the bloodstream after t days,

$$druglevel(t) = 60(.75)^t \qquad (5)$$

$$\text{Domain}: \{t \mid 0 \le t \le 50\} = [0, 50] \text{ (after 50 days amount of drug is negligible)}$$
$$\text{Range}: \{druglevel \mid 0 \le druglevel \le 60\} = [0, 60].$$

Therefore, the drug after each 24-hour time period can be determined by

$$druglevel(1) = 60(.75)^1$$
$$druglevel(2) = 60(.75)^2 = 60(.75)(.75)$$
$$etc...$$

***Question 7** *Using Equation (5), modeling the drug level in the bloodstream, determine how much remains at time=3, 4, and 5 days. Determine how many milligrams of Prozac your body metabolizes between each successive day for days 0 to 5. What do you notice about the differences between the previous and successive drug levels?*

***Question 8** *Utilizing the function above that models the amount of Prozac in the bloodstream, complete the following:*

a. *Find the amount of Prozac in the bloodstream after one week.*
b. *Estimate the half-life (the amount of time required to decrease the original amount by one half) of Prozac in the bloodstream.*
c. *Estimate how long it takes until the level drops to 10mg.*

***Question 9** *Equation (5) provided a function that yields drug level as a function of time. You were able to iterate to find the answer to Question 7. What other technique could we use to determine the day at which Prozac level reached 10mg? What answer did you get?*

Question 10 *Under ideal conditions, bacteria flourish in a given cadet's sink. However, it is Tuesday afternoon and you are preparing for WAMI. There are currently 1000 bacteria in your sink. Your friend from another regiment measured how bacteria remained in his sink after he sprayed it at 2400 hours the night before his last WAMI. Below is the table that he made.*

Hour	Bacteria
0	1000
1	500
2	250
3	125
4	62.5
5	31.25

a. How many bacteria would be left after 8 hours?

b. Given an inspection that starts at 0630, when would you have to spray to ensure there were no living bacteria in your sink at the beginning of inspection?

c. Your squad leader is a Chemistry major. He begins to inspect your room and sees that there are 5 bacteria remaining in your sink. When did you spray?

Question 11 *Go to the web or any other reference and find the population data of your favorite country (other than the US). Plot the data, and predict the population for the years 2010, 2020 and 3000. What type of model did you use? What assumptions did you make? Discuss how good you think your predictions are and why?*

Question 12 *The cost of a US first-class postage stamp was 29 cents in 1990 and was 39 cents in 2006; predict when the cost of this stamp will be $1 using an exponential model. **HINT** You may want to scale the years so you are not raising a number to the 1990 power.*

Question 13 *The population in Orange County, New York in 1990 was 307,647 and increased to 341,367 in 2000. Using this information and assuming an exponential model, what do you predict the population will be in 2010?*