1.8 Generalized Power Function

Power functions can be extremely useful models that describe many interesting phenomena in our world. Shortly after the apple fell on his head, Sir Isaac Newton discovered that gravity could be represented and modeled using a power function, namely a quadratic function of the form $y=x^2$.

Recall our discussions of rates of change in sections 1.3 and 1.6. Up until now, if the nature of the data we were presented with demonstrated a constant rate of change, we could use a two-parameter linear model. If the data have curves in them (non-constant rate of change), we could use a three-parameter exponential function to model the data. In this section, you will learn about another model that is useful in modeling data with curves – the four-parameter power function.

In the linear case, we saw that the rate of change was constant and its sign determined if the line was increasing or decreasing. We will see many examples of power functions that increase or decrease and their rates of change that also increase or decrease. Figure 1.42 catalogs the four possibilities of power functions and their corresponding changing rates. Each of the pieces of the black circle represents a piece of a function. The adjacent box explains the behavior that is graphically depicted by the corresponding segment, i.e., how the function and its rate of change are changing. Use this diagram as a tool to help analyze new functions throughout this text. **NOTE** This is the same figure as Figure 1.26 in Section 1.6 on exponential functions.

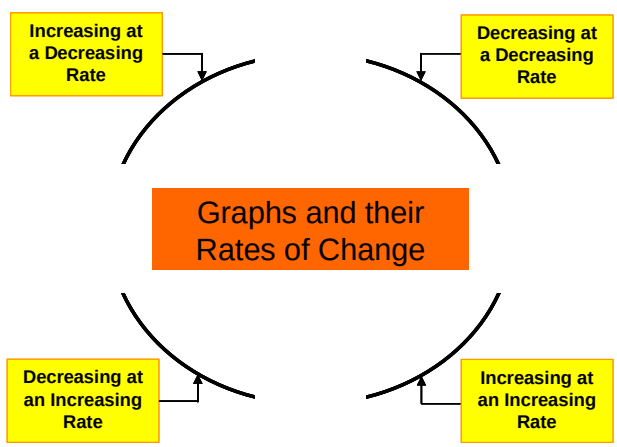

Figure 1.42: Graphs and rates of change.

1.8.1 Three Cases of Power Functions

We will consider three different cases of the power function

$$f(x) = x^b \qquad (1)$$

where x is the independent variable and b is a parameter that determines the shape of the power function. Notice there is a distinct difference between the

shapes of the graphs when b is even or odd. We will show each of these two standard functions in each of the three cases. There are many forms of the power function other than the three that we will discuss, but all power functions are characterized by the *independent variable* being raised to a constant power.

Case 1 $b = n$, **where n is a positive integer**

The parabola

$$f(x) = x^2$$

is a power function. As characterized above, our independent variable is raised to a constant power (in this case, the power is 2).

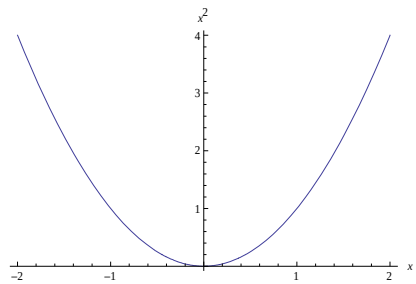

Figure 1.43: Plot of Power Function x^2

Another example of Case 1 is the cubic function

$$f(x) = x^3$$

which has parameter $b = 3$.

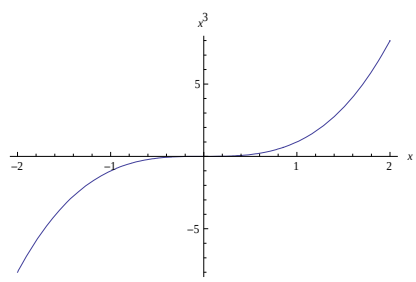

Figure 1.44: Plot of Power Function x^3

Note that if $b = 1$ we have the function $f(x) = x$, which is linear. Therefore, the linear function is a particular type of power function.

Question 1 *Plot x^4 by hand. How do the plots of x^2, x^3, and x^4 compare to each other? What can you conclude about the effect of the b parameter on Equation 1?*

Question 2 *What are the domain and range of functions that fall into Case 1?*

Question 3 *What can you say about the rates of change of the functions in Case 1 ?*

Question 4 *The kinetic energy (KE) from a moving billiard or pool ball can be measured by the relationship:*

$$KE = \frac{1}{2}mv^2$$

a. *Given a constant mass, what does doubling the velocity (v) do to the amount of kinetic energy?*

b. *How could you describe the rate of change of this function?*

An example of a function in Case 1 is the relationship for the area of a circle with radius *r*, $A = \pi r^2$. This relationship is an example of a power function where *b*=2, the radius, *r,* is the independent variable, and *A* or *A(r)* is the dependent variable.

Thus far, we've considered the case where our exponent is a positive integer. We will discuss the two other cases that may prove useful in modeling.

Case 2 $b = 1/n$, **where *n* is a positive integer** (e.g., $f(x) = x^{1/2} = \sqrt{x}$)

This type of power function is frequently referred to as a **root function.** *Root functions* involve taking a root of the *independent variable* and exhibit one of the two general shapes shown in Figures 1.45 and 1.46.

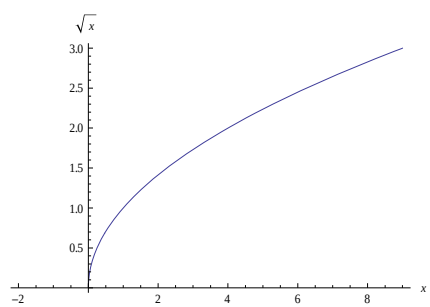

Figure 1.45: Plot of Power Function $\sqrt{x} = x^{\frac{1}{2}}$

Power functions whose n values are even numbers like 2, 4, and 6 are called **even root functions**. The domains of even root functions are limited to $[0, \infty)$.

Question 5 *How do the plots of* $\sqrt{x}$, $\sqrt[4]{x}$, *and* $\sqrt[6]{x}$ *compare to each other? What are your conclusions with respect to their rates of change?*

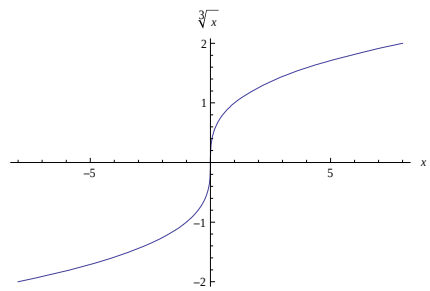

Figure 1.46: Plot of Power Function $\sqrt[3]{x} = x^{\frac{1}{3}}$

Power functions whose n values are odd numbers like 3, 5, and 7 are called **odd root functions**. The domains of odd root functions consist of all real numbers.

Question 6 *How do the plots of* $\sqrt[3]{x}$, $\sqrt[5]{x}$, *and* $\sqrt[7]{x}$ *compare to each other? What are your conclusions with respect to their rate of change?*

Case 3 $b = n$, **where *n* is a negative integer**

This is known as a **reciprocal function**. Its characteristic shape is a hyperbola as shown in Figures 1.47 and 1.48.

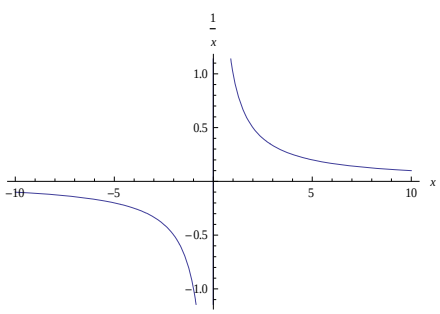

Figure 1.47: Plot of Power Function $\frac{1}{x} = x^{-1}$

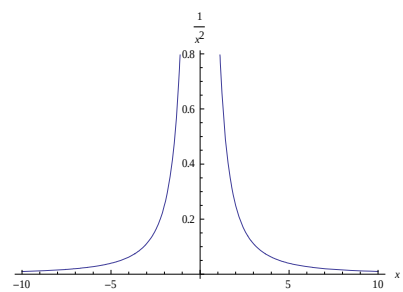

Figure 1.48: Plot of Power Function $\frac{1}{x^2} = x^{-2}$

Each of these two graphs have horizontal and vertical asymptotes that separate it from the previous cases. The vertical asymptote is the *y*-axis or the line $x = 0$.

The horizontal asymptote for each of these functions is the *x*-axis or $y = 0$.

Question 7 *What are the domains and ranges of Case 3 power functions?*

Question 8 *How do the plots of* $\frac{1}{x}$, $\frac{1}{x^3}$, *and* $\frac{1}{x^5}$ *compare to each other? What can you say about the rates of change of these functions on the different intervals in their domains?*

Question 9 *How do the plots of* $\frac{1}{x^2}$, $\frac{1}{x^4}$, *and* $\frac{1}{x^6}$ *compare to each other? What can you say about the rates of change of these functions on the different intervals in their domains?*

This function below is often used with a limited domain to describe real world phenomena. It is Boyle's Law from chemistry and physics which states that for a constant temperature, *C*, the volume of gas, *V,* is inversely proportional to the pressure, *P.*

$$V(P) = C^{\circ}P^{-1} = \frac{C^{\circ}}{P}$$

Question 10 *Using Figure 1.42, the circle diagram, give an example of a power or polynomial function and its domain that satisfies each of the four cases.*

1.6.3 The Generalized Power Function

Regardless of the type of function we're working with (our readings have covered the *linear, exponential, and power families),* functions contain *parameters* or constant values that influence the shape and location of their graphs. Experimentation with graphing in *Mathematica* should have resulted in some generalities regarding the *b* parameter for power functions. However, most data that follows a power function trend does not go through the origin, so we find modeling with the form in equation (1) to be too restrictive. We use the more general form

$$f(x) = a(x+c)^b + d$$

where *a*, *b, c*, and *d* are parameters that can assist us in modeling data that follows a power trend. We call this function the *generalized power function.*

Question 11 *Begin with the function* $f(x)=ax^2$. *Select a positive and negative whole number and decimal value for the a parameter. What effect can you conclude that this parameter has upon the generalized power function?*

Question 12 *Begin with the function* $f(x)=(x+c)^2$. *Select a positive and negative whole number for the c parameter. What effect can you conclude that this parameter has upon the generalized power function?*

Question 13 *Begin with the function* $f(x)=x^2+d$. *Select a positive and negative whole number for the d parameter. What effect can you conclude that this parameter has upon the generalized power function?*

When b is a non-negative integer (Case 1), then the power function $f(x)$ is also called a polynomial. A **polynomial** function is a linear combination of power functions with non-negative integer powers. They can all be written in the form:

$$f(x) = a_b x^b + a_{b-1} x^{b-1} + \ldots + a_2 x^2 + a_1 x + a_0$$

where b is a non-negative *integer*. The **coefficients** or numbers appearing before each term, $a_0, a_1, a_2, \ldots, a_b$, are constants within the polynomial.

For example,

$$f(x) = 3(x-6)^3 + 4 \;=\; 3x^3 - 54x^2 + 324x - 644$$

after expanding the polynomial. So, the coefficients of the polynomial are $a_3 = 3, a_2 = -54, a_1 = 324$, and $a_0 = -644$. However, not all polynomials can be written in the generalized power function form. For example,

$$f(x) = x^3 + x^2 + x + 1$$

cannot be written in the form $f(x) = a(x+c)^b + d$. There are many applications that can be modeled using polynomials, but we will restrict our work to this generalized power function for polynomials. We do this because the general polynomial

$$f(x) = a_b x^b + a_{b-1} x^{b-1} + \ldots + a_2 x^2 + a_1 x + a_0$$

has $b+1$ parameters and would require $b+1$ points to find each parameter. Solving for so many parameters is time consuming and often leads to low predictive power (recall our discussion in section 1.5).

1.8.4 The Role of Parameters

Understanding the role each parameter plays in our families of functions contributes to our ability to create useable *mathematical models*. Initially, understanding function *parameters* assists in the selection of an appropriate function family. Next, knowledge of *parameters* enables us to improve our *mathematical model* which ultimately results in a model that can be used for prediction.

When modeling using a power function, the first step is to choose an appropriate b value based on the general shape of the curve. After estimating or assuming b, it is important to estimate the other parameters. First, consider the role of the d parameter. For linear functions, it describes the y-intercept. If we assume that $x = 0$ in the generalized power function, we get $y = ac^b + d$, so d is not the y-intercept for a generalized power function (unless a or c is also 0). Look at the graphs in Figures 1.49 through 1.51 to see how the d parameter changes them.

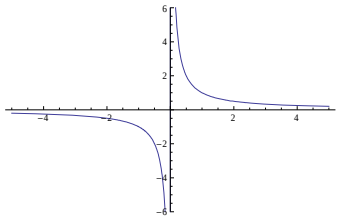

Figure 1.49: Plot of Power Function $\frac{1}{x}$

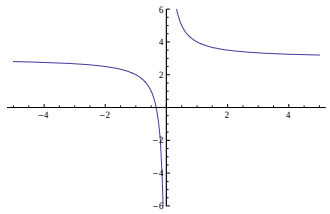

Figure 1.50: Plot of $\frac{1}{x}+3$

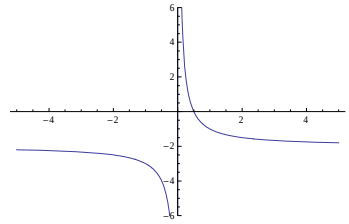

Figure 1.51: Plot of $\frac{1}{x}-2$

Generalization 1. We can shift any graph of a function upward or downward by adding a parameter to the function:

$$y = f(x) + d$$

If d is positive, we will shift or translate the function's graph upward. If d is negative, we will shift or translate the function's graph downward. ****NOTE**** This is also the same first generalization we made for the linear and exponential functions.

Next, consider the c parameter. In Figures 1.52-1.54, we see that this parameter creates a horizontal shift as opposed to the vertical shift of the d parameter. In these figures, $f(x)=\sqrt{x}$, $f(x+2)=\sqrt{x+2}$, $f(x-3)=\sqrt{x-3}$. You may expect that a positive value of c will move the function to the right and a negative to the left. Notice that the actual shift is opposite: positive c values shift the function left and negative c values shift the function right.

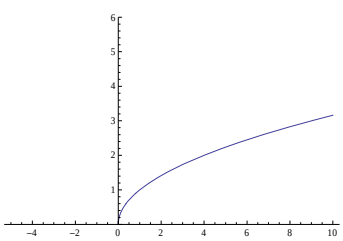

Figure 1.52: Plot of $\sqrt{x}$

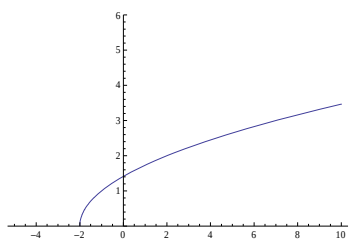

Figure 1.53: Plot of $\sqrt{x+2}$

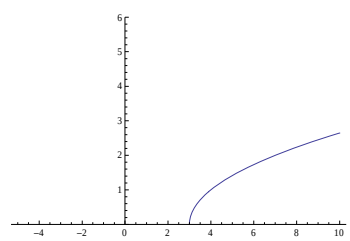

Figure 1.54: Plot of $\sqrt{x-3}$

Generalization 2. We can shift any graph of a function to the right or left by adding a parameter to the function as:
$y = f(x+c)$

If c is positive, we will shift or translate the function's graph to the left. If c is negative, we will shift or translate the function's graph to the right.

Estimating the a parameter in $f(x)=a(x+c)^2+d$ is not as easy as the other three parameters. We often estimate the b parameter based on shape (s-curve versus u-shaped, ...), then estimate the c and d parameters using horizontal and vertical shifts, respectively (often using the coordinates of the first data point). We can then solve for a using a representative point from the data that is different from the first data point.

Question 14 *What happens if you change the 'a' parameter? Like in the above examples, experiment with different values using Mathematica.*

Example 1 Case 1 of the Power Function. *Suppose a paratrooper jumps from a helicopter at a height of 1000 meters. During his descent, he radios you and reports altitude. Five seconds into his descent, he is at an altitude of 878 meters.*

Sir Isaac Newton described freefall motion by a quadratic (raised to the second power) power function. Using the data provided by the paratrooper, determine the specific power function that describes the paratrooper's descent.

REAL WORLD PROBLEM: Model the paratrooper's descent using a generalized power function.

Step 1. Transform problem into a mathematical model.

a. **Given:** Generalized power function: $Alt(t) = a(t+c)^b + d$, $b = 2$
The points $(0,1000)$ and $(5,878)$.
Variable declaration: $Alt(t)$ is the altitude of the paratrooper after t seconds.

b. **Find:** Parameters *a, b* (given)*, c, d:* the final model of *Alt (t).*

c. Solution **Plan**: Use parameter estimation techniques to find *b, c*, and *d* and then use a representative point to find *a*.

Step 2. **Solve** using appropriate solution techniques (algebraic manipulation).

A good estimation for the *c* and *d* parameters are the *x* and *y* coordinates of the first data point, respectively. Therefore, assume $c = 0$ and assume $d = 1000$. Our new function looks like

$$Alt(t) = a(t+0)^2 + 1000 = at^2 + 1000.$$

To solve for *a*, we use another representative data point. Let's use the second point $(5,878)$.

$$Alt(5) = a(5s)^2 + 1000m = 878m$$

$$a = -4.88\frac{m}{s^2}$$

$$Alt(t) = -4.88t^2 + 1000$$

The domain of this model has a time restriction. Because it doesn't make sense to model earlier than our first recording, we will limit the lower bound of the independent value to zero seconds. Furthermore, we need to limit the upper bound on our domain value because eventually the jumper will hit the ground. Once this happens, time will continue but our model will no longer make

reasonable predictions. To see this, let's try it. What if we wanted to see what our model predicts 60 seconds into the paratroopers flight?

$$Alt(60) = -4.88(60)^2 + 1000 = -17568 + 1000 = -16568m$$

So, for what range of t should our model be valid? It should work until the jumper hits the ground or has an altitude of zero. Using this information, we can algebraically solve for the time when this happens.

$$0 = -4.88(t)^2 + 1000$$

$$4.88t^2 = 1000$$

$$t^2 = 204.82$$

$$t = \pm 14.315 \sec$$

Where does this leave us? We previously established that the lower bound of our domain is zero, eliminating the -14.315 seconds. Therefore, our domain consists of all times between 0 and 14.315 seconds and our range consists of values between 0 meters and 1000 meters.

Step 3. Communicate and **reflect** upon results.

The paratrooper fell 1000 meters in 14.315 seconds according to our model. This means that the paratrooper fell at nearly 70 meters per second on average or about 157 miles per hour. This may seem fast, but this is close to the falling speed due to gravity.

Question 15 *What luck! It turns out the paratrooper's altimeter is digital and it recorded his jump. The collected data are in Table 1.13 and are linked on the course website. Using two new data points develop another function modeling the paratrooper's decent. How does your new model compare to your old? Which is better? Why?*

Time (sec)	0	1	2	3	4	5	6	7	8	9	10
Altitude (m)	1000	995.01	978.57	953.6	917.3	873.96	810.77	741.92	660.8	576.5	456

Table 1.13: Altimeter Data on Paratrooper Jump

Example 2 Case 2 of the Power Function. *Data have been collected on weightlifters' body weights and their corresponding squatting, benching and dead-lifting strengths. We'd like to model weightlifter benching strength as a function of body weight. The data is provided in Table 1.14 and are linked on the course website.*

Weight	Squat	Bench	Deadlift
114	330	180	415
123	415	245	425
132	495	285	540
148	535	335	584
165	610	390	660
181	670	473	725
198	700	500	730
220	722	510	705
242	720	529	755
275	730	525	700

Table 1.14: Weightlifting Data

REAL WORLD PROBLEM: Model Weightlifting data

Step 1. Transform problem into a mathematical model.

a. **Given:** Data on weightlifters ranging in weight from 114 to 275 pounds. A plot of the data in Figure 1.55 suggests the shape of an inverted parabola or perhaps a root function.

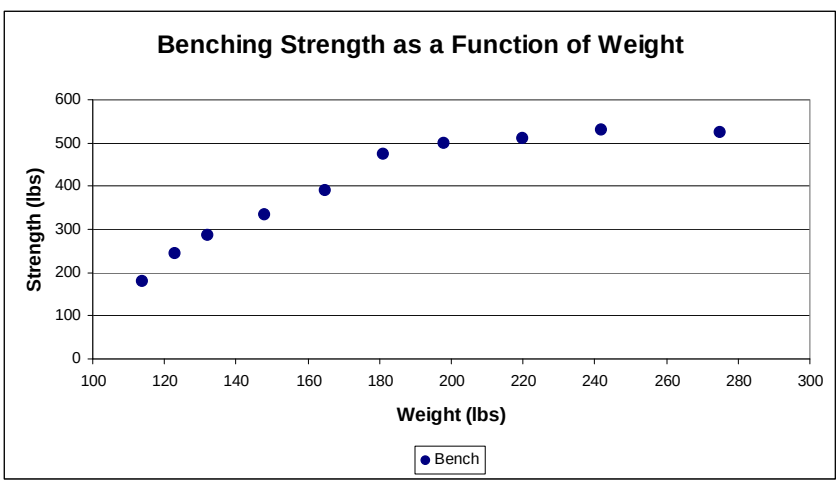

Figure 1.55: Graph of Weightlifting Data

b. **Find:** We are trying to model how much a weightlifter can bench given his or her body weight. In essence, we would like to use our independent variable, *Weight,* to predict our dependent variable, *Benching Strength.*

c. **Assume:**

Question 16 *What must we assume to solve this problem?*

d. Solution **Plan**: Since we are trying to model *Benching Strength* as a function of *Body Weight, Body Weight* is the independent variable and *Benching Weight* is the dependent variable. In general, because our data exhibit a curve that increases at a decreasing rate, our model will be:

$$Strength(Weight) = a(Weight + c)^{\frac{1}{n}} + d$$

Next, we'll need to estimate values for the parameters (a, $b = 1/n$ c, and d) of our *root-function power model*. We can use three representative points to find the a, c, and d parameters after making an assumption about the b parameter. Because we know the b parameter must be between zero and one, let's assume $b = \frac{1}{2}$.

Step 2. **Solve** using appropriate solution techniques (algebraic manipulation).

We assumed that $b = \frac{1}{2}$, now we need three representative data points to estimate the a, c and d parameters. Using the second, fifth and eighth data points yield the following three equations:

$$245 = a(123 + c)^{1/2} + d$$

$$390 = a(165 + c)^{1/2} + d$$

$$510 = a(220 + c)^{1/2} + d$$

Thus, after using *Mathematica* to solve for a, c and d, our model is

$$Strength(weight) = 39.6363(weight - 107.702)^{1/2} + 89.9702\,.$$

Its domain and range are:

$$Domain: \{weight \,|\, 114 \le weight \le 400\} = [114, 400]$$

$$Range: \{strength \,|\, 180 \le strength \le 785.359\} = [180, 785.359]$$

Certainly, these numbers are not exact but we are trying to identify that our model has limitations. Here, we are saying that we believe our model is valid for individuals weighing between 114 and 400 lbs since those weighing less or more are unlikely to be lifters. Also, we are saying there are limits on the amount a person can bench which depend on our domain constraints. Perhaps you feel this upper value is high and would like to see it be lower. That's ok.

Sometimes models don't have exact bounds on their domains and ranges or in the course of study you realize revision to your initial estimates is necessary.

Step 3. Communicate and **reflect** upon results.

Our calculations show that our model predicts our second, fifth and eighth data points. This should be the case since these are the three data points we used to solve for the parameters. Notice that the model goes through these points in Figure 1.56.

Now that we have developed a model, we can use it to verify weightlifting strength within our data set; recall that we referred to this as *interpolation*. Often, a model which predicts well within the data is also useful outside of the collected range of data; recall that we referred to this as *extrapolation*. Therefore, we should use our model to predict a data point within the data and also one outside of the data to test how useable our model appears to be.

$$Strength(165) = 39.6363(165 - 107.702)^{\frac{1}{2}} + 89.9702 = 389.999$$

$$Strength(350) = 39.6363(350 - 107.702)^{\frac{1}{2}} + 89.9702 = 706.946$$

Our model appears to predict fairly well for the given data but appears too high for prediction outside of the range of our data. Figure 1.56 is a plot of our data with our model overlaid on the same axes.

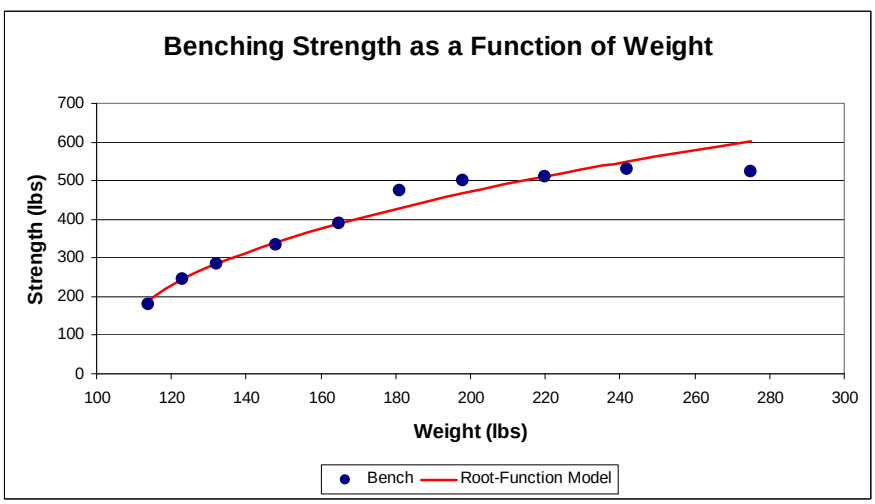

Figure 1.56: Weightlifting Problem – Plot of Model and Data

Question 17 *Based on your weight, what does this model expect you to bench?*

Question 18 *Based on the amount you can bench, what does this model predict you should weigh?*

Question 19 *Do a sensitivity analysis on our assumption that* $b = \frac{1}{2}$ *to see how this affected our model. Is the new b value a better representation of the data?*