UNITED STATES MILITARY ACADEMY

FISHIN FOR RESULTS

146/150

MA 205: CALCULUS II

SECTION C 5

LTC MEYER

Exellent

A+

BY

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WEST POINT, NEW YORK

11 MAY, 2008

_______ MY DOCUMENTATION IDENTIFIES ALL SOURCES AND ASSISTANCE RECEIVED IN COMPLETING THIS ASSIGNMENT.

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SIGNATURE: J. Puttmann Ezra Swanson

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INTRODUCTION:

After an airborne re-supply mission went wrong and our pallet crashed, soldiers recovering the pallet noticed that the fish in a pond nearby the crash had all died. While conducting the analysis it was apparent that the wildlife that typically uses the pond as a source for water was staying away.

During the inspection of the pond one of our soldiers noticed a floating a container and managed to pull it out. Looking at the container it be~~comes~~ ame apparent why the fish population in the pond ~~is~~ was dead. The label on the container read "ACETYLENE DICHLORIDE" with a concentration of 1000 mg/gallon. Re-checking the load list we notice that there were 3 of these containers in the pallet. Unfortunately the other two containers are unaccounted for. Back at the base we contacted our battalion chemical officer about the container and he explains that everything needs to be flushed with fresh water and that the pond should not be used in anyway by locals. After looking deeper into the particular chemical, we determine that the pond will not become livable for fish until the chemical level drops below 10 parts per million (ppm) where 1 ppm = $3.97\,mg/m^3$. We find our situation is worsened by the fact that there is a village near by that depends on the pond for fish to sustain them.

Stay in past tense

We conduct a survey in order to determine how much fish the village depends on per day so we can expediently get the pond back to normal so the villagers can begin fishing again. Our survey indicated that the villagers take 10 fish per day at about 2lbs each for a total of 20lbs. Based on our count of dead fish we determine that at the time of the contamination, the pond sustained a fish population of 750. We contacted a local fish hatchery that raises the same species of fish to re-stock the pond. The hatchery reported that they started with 20 fish and 14 days later the number had grown to 45. The species is known for quick reproduction and growth so such a large growth rate was normal. Until we can re-populate the pond with the correct number of fish, the military makes up for the lost fish production by supplying the village with food at a cost of $100 dollars per day.

METHODOLOGY:

Here we incorporate the information gathered in our survey and our analysis of the pond to determine when the pond can become livable and the amount of fish with which to re-populate the pond so that the population can sustain itself. We also need to minimize the cost to do so. Before we can begin our mathematical representation and findings for this problem there are a few facts that need to be stated and assumptions we need to make.

Facts:

- Flow Rate into the pond is 5 gallons/min
- Flow Rate out of the pond is 5 gallons/min
- Every day the villagers cannot fish costs the US Gov. $100 in supplemental food
- Each fish purchased from the hatchery costs $10
- Pond will not sustain fish until chemical level drops below 10 *ppm*
- $1\ ppm = 3.97\ mg/m^3$

Assumptions:

- Pond depths are accurate as well as surveys and analysis.
- Flow rates in and out of the pond are accurate and do not change this time of year. (not rainy season which would change flow rates)
- Fish hatchery has 100's of fish available

- Contents of all three chemical containers are in the pond.
- Fish from hatchery are extremely adaptable and healthy.
- Villagers maintain there previous rate of 20lbs of fish per day.
- Fish from hatchery provide the same amount food per fish as the previous fish.
- After chemicals leave the pond they will not have any other adverse effects on the village.
- The fish are not leaving the pond in any other manner than the villagers fishing. And if they are it is already accounted for in our logistics model.

Once our assumptions have been made we break the problem up into individual portions. The first part of the problem ~~needing to be solved~~ is to determine when the pond will drop below 10 ppm. This is best done by seeing how long it will take the water flowing into the pond to flush the chemicals out of the pond and into the ocean where it will become harmless.

It is important and intelligent to assume that all the containers contents are in the pond, because if there is less it will only mean we are extra safe. We could assume that only the one container we found dumped chemicals in the pond, but if we were wrong all the new fish we put into the pond would die. We may lose some money due to "over-flushing" the pond, but assuming this max amount in the lake reduces risk in the long run.

The next step in our problem is to find how many fish we need to introduce into the pond so that the number taken by the villagers per day won't deplete the population and cost the defense department more money in supplementary food. This is best done with a logistics model which accounts for the carrying capacity of the pond, and the growth rate of the fish.

When this is complete through various assumptions and our results from parts 1 and 2 we can determine the cost to re-stock the pond so that the fish can self sustain and the defense department does not have to spend money again to re-stock or supplement the village. We also elected to do a sensitivity analysis in order to see what the cost would be to either re-stock the pond with as few fish as possible and the cost to do so, as well as how long it would take the pond to drop below 10 *ppm* should we assume that the only container of chemical that was dumped into the pond was the one we accounted for.

Excellent

FINDINGS:

First, we want to know when the pond will become livable. We do this by finding the volume of the pond using the upper right hand rule because we are provided a contour plot complete with the different depths of the pond. We find that the pond's volume is 14,000 cubic ft where 1 cubic foot = 7.48051948 US gallons so the total PondVolume equal to 104717 gallons. We then need to find the concentration of chemical in milligrams. We know that there are between 30 and 90 gallons of chemical in the pond. That would mean based on the metric conversion there are between 30,000 and 90,000 mg of chemical in the pond.

Explain this more.

We know that before fish can live in the pond the level of concentration has to drop below 10 ppm of where 1 $ppm = 3.97\, mg/m^3$. We also need to know that 1 meter cubed is equivalent to 264.172052 US gallons. The minimum chemical volume can found with this equation; ($MinChemAmount = PondVolume * \frac{1}{1000000} * 3.97 * 264.172$). We find the $MinChemAmount$ = 15738.5 mg in the pond. Next we need to define our variables.

We define all our necessary variables and initial conditions so we can reference them with each portion of the problem we are trying to solve. This portion includes c[t] as the amount of chemical in milligrams at any time 't' in minutes. c'[t] is the rate of ~~water~~ mg chemical in the pond, minus the rate of ~~water~~ mg out of the pond. We utilize the DSolve command in Mathematica using our initial condition and recalling all our previously defined equations and variables. This gives us the amount of time, 't' in seconds that are then converted to hours, and then days it takes the stream to flush out the chemical in the pond so that the fish can once again survive in the pond. (Reference Appendix A) We find that assuming all the chemicals in the pallet are now in the pond, it will take a total of 25.4 days before we can introduce the new population of fish into the pond. Because we will have already paid for the village's food on the 25^{th} day, we elect to introduce the fish on the morning of the 26 so we can avoid paying the $100 dollars for the 26^{th} day.

Now that the question of when the fish can be introduced to the pond has been answered, we can begin finding how many we need to purchase so that the population can self sustain while still sustaining the village needs. We do this by again referencing our previously defined variables. We know that the best model for the growth of the fish is a logistics equation because it accounts for the carrying capacity of the pond. The general form of a logistics equation is:

$$\frac{dP}{dt} = k * P(1 - \frac{P}{K})$$

In this equation, k represents the constant rate at which the fish produce, P represents the population and K represents the carrying capacity of the lake. First, we must first find the proportionality constant k. We use the information of their pervious growth rate to determine this. (Reference Appendix B)

We determine the value of k to be 0.0579236. Once k has been determined we then plug k into our logistics model and use DEPlot command in Mathematica, where, P[t] will represent the population of the fish at anytime't' and P'[t] represents the rate of change of the fish population. The DEPlot command will show what type of behavior is being displayed by the fish population in the lake. This will also show where the population equilibrium is and more importantly the initial amount of fish that must be introduced to the pond so the population can maintain itself while being taken by the villagers. (Reference Appendix C)

Once the behavior of the population has been modeled, we then use the Solve command in Mathematica to find the specific values pertaining to the equilibrium and minimum amount of fish that must be introduced in order to reach that equilibrium. Once the equilibrium and minimum amount of fish have been found, we then find out how many days it will take to reach that self sustaining population with the minimum amount of fish.

We determine the equilibrium population to be 779 fish and an initial population of 230 will reach self sustainment in 40 days which includes fishing done by villagers the same day the fish are re-stocked assuming only ten fish are removed each of the 40 days. Any number of fish purchased below 230 will take time to grow to the stable population and cost the defense department more money in supplemental food. (Reference Appendix B) Utilizing this same equation we can also find out how long it will take for the fish population to double. If we set the initial condition to 230 and the population to 460, we find it takes 18 days for the population to double. Example:

```
In[101]:= Solve[p[t, 230] == 460, t]
Out[101]= {{t → 18.0923}}
```

With all this information taken into account we can then analyze the best cost efficient way to re-populate the pond so that the villagers can once again begin fishing the lake and the defense department can stop supplementing the village food at the cost of \$100. To find the best cost efficient and safe way to re-stock the pond, we manipulated our growth model and included our costs of both supplementing the village food, and price per fish. (Reference Appendix D)

DISCUSSION OF RESULTS:

Based on our findings, we were able to determine that the pond will become livable after 25.5 days. On the morning of the 26th day since our failed airborne re-supply mission we will introduce the fish. We determined that the minimum number of fish needed to be introduced into the pond to be 230. The villagers may begin fishing immediately as our logistics model accounts for the 10 fish the village had been harvesting per day prior to contamination. If the villagers maintain this rate of fish caught for food, the pond will reach its equilibrium population in 40 days.

Based on these numbers, and the fact that the army is extremely sensitive to the needs of these villagers, we believe it would pay large dividends with the village if we can re-populate the fish as soon as possible. Because the pond is not livable for 25 days, the defense department will have to supplement the village food for that time at the cost of \$100 per day. Regardless of the number of fish we purchase we know that we are going to spend \$2500 automatically. We can save money by avoiding extra days supplementing the village food. To find the total cost of the re-population project, days supplemented must be added to the cost of fish. If we purchase the minimum number of 230 fish at the cost of $10 we will spend $2300. Adding both leaves the defense department with a total cost of $4836.29. This number is slightly higher than simply adding cost of fish to the cost of food, because our mathematical models and equations contain decimals. (Reference Appendix D)

SENSITIVITY ANALYSIS:

The purpose of conducting a sensitivity analysis is to have something to compare our findings to by changing an aspect of our initial assumptions. The first assumption we want to examine is the amount of chemical that was accidentally dumped into the pond. We initially solved this problem assuming that all 90 gallons of chemical missing from the pallet was introduced into the pond. We made this assumption to be as safe as possible and to avoid having to re-stock the pond more than once and because 60 gallons of the chemical was unaccounted for. We now want to assume that perhaps later in our clean up we are able to account for the missing 60 gallons and discover that it had not been introduced into the pond. This would mean we have to find out how much time it would take for the pond to clean 30 gallons out of itself until it reaches below 10 ppm. We do this utilizing the same procedure as if the pond had 90 gallons. Recall the equation to solve for the minimum chemical volume,

($MinChemAmount = PondVolume * \frac{1}{1000000} * 3.97 * 264.172$). We find the

$MinChemAmount$ = 15738.5 mg in the pond. We then Solve for c'[t] which like the first portion of our problem, is the rate of water in the pond minus the rate of water out of the pond. We utilize the DSolve command in Mathematica using our initial condition and recalling all our previously defined equations and variables. (Reference Appendix D)

Our Mathematica gives us a ‘t’ value in seconds which we convert to hours and then to days. We find that if we only assume 30 gallons are in the pond it will take 9.4 days before fish can be introduced into the pond safely.

Assuming that we want to let the villagers fish as soon as possible and with our knowledge of how the fish population re-produces we know that there must be 230 fish introduced. With that knowledge and the cost of the fish we know that if we only assume 30 gallons of chemical are introduced into the pond, that the cost to re-stock the pond is $3238.3, a difference of $1597.99, from assuming all 90 gallons are in the pond. Looking at the potential cost we believe the assumption that all 90 gallons are in the pond is paramount to ensuring that the defense department only has to re-stock the pond one time as there is less risk of the initial re-stock population dying

The next aspect of our sensitivity analysis is that the fish hatchery does not have more than 10 fish available for purchase. This means that the defense department is going to have to supplement the village food until the population reaches 230. We find out how many days it will take for the population to reach this number by solving for the population with respect to time, given our initial condition of 10 fish. We utilize Mathematica and our previous variables and equations to do this. (Reference Appendix F) We find that it would take 59 days for an initial re-stock population to reach 230 fish at which time the villagers could begin fishing. With the cost of supplementing the villagers’ food and the cost to purchase the initial 10 fish, the defense department would end up spending $8,536.29 on the re-stock project, assuming the 90 gallon clean up time. (Reference Appendix C & D)

Predators may also utilize the ponds fish population as a source of food so the extra 7 fish should account for any old and dying fish as well as fish taken by predators because. We believe this buffer to be important because 1 fish below 222, in addition to the villagers fishing, would deplete the population and the defense department would have to again supplement village and re-stock the pond. Because of this we elected to round the initial re-stock population up to 230.

CONCLUSION:

While working on the most cost efficient way to re-populate the fish population it is important to be continuously reminded that the chemical in the pond has to be below 10 *ppm* or else any fish in the pond will die. In actuality our math says that the pond drops below this amount after about 25.5 days. Because we have already paid for the village to be supplemented the 25 days of food, we thought it an intelligent decision to wait the extra hours allowing the pond's level of contamination to drop even further below 10 *ppm* to a lesser number as time was not going to affect our cost to supplement the village for the 25th consecutive day.

Our sensitivity analysis validates the course of action that we aim to take. By changing how much chemical we assumed entered the pond and how we wanted to go about purchasing the fish to repopulate the pond, we demonstrated that our course of action minimizes the risk at an affordable cost.

If we re-stock the pond on the morning of day 26 we can save $100 as the villagers can begin fishing that morning. The number of fish needed to re-stock the pond was in actuality 222 fish, however we decided to go with 230 as we can not be 100% sure that all 222 fish would survive. The cost analysis had to factor in time. Knowing the pond is unable to sustain fish until day 25 and it is more expensive to supplement the village food, we elected to purchase 230 and re-stock the pond on the morning of day 26. They can begin fishing that day while not damaging the fish population. We recommend the villagers be supplemented food by the US government for 25 days, and on the 26th they can begin fishing at a total cost of $4836.29.

Excellent.

Appendix A

▪ Time to flush the pond if 90 gallons are spilt.

In[31]:= Vc2 = 1000 * 90 "mg in pond"

Out[31]= 90 000 mg in pond

The first thing we need to do is define our variables. c[t] is the amount of chemical in milligrams in the pond at time 't' in minutes. c'[t] is the rate in minus the rate out. Here, using our c'[t] equation we find the c[t] using the initial condition. **needs more explanation.

In[32]:= $\text{DSolve}\left[\left\{c'[t] == -5 * \frac{c[t]}{\text{PondVolume}},\ c[0] == \text{Vc2}\right\},\ c[t],\ t\right]$

Out[32]= $\{\{c[t] \to 90\,000 \text{ mg in pond } e^{-0.0000477431\,t}\}\}$

In[33]:= $c2[t\_] = 90\,000\ e^{-0.00004774305555887104`\ t}$ "mg in pond"

Out[33]= $90\,000 \text{ mg in pond } e^{-0.0000477431\,t}$

In[34]:= Solve[c2[t] == MinChemVol, t]

Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. »

Out[34]= $\left\{\left\{t \to -20\,945.5\ \text{Log}\left[\frac{0.0000111111\ \text{MinChemVol}}{\text{mg in pond}}\right]\right\}\right\}$

In[35]:= 36 522.6 / 60 "hours"

Out[35]= 608.71 hours

In[36]:= DaystoClean90G = 608.7099999999999` / 24

Out[36]= 25.3629

Appendix B

▪ Finding number of fish needed to restock.

Using a the given information and a logistics equation we can model the fish population. We first model the population of fish without any effect from the villagers or environment.

```
In[65]:= << DiffEqs`DEGraphics`

Emmigration = 10 "fish per day"

10 fish per day
```

If the initial amount of fish is 20 they grow to 45 after 14 days. Here we find the differential equation that gives the amount of fish after time 't'. p'[t] is the rate of change of the fish population and p[t] is the amount of fish at a certain time 't'.

```
DSolve[{p'[t] == k*p[t], p[0] == 20}, p[t], t]
```

$\{\{p[t] \to 20\, e^{0.0579236\, t}\}\}$

```
p1[t_] = 20 E^(k t)
```

$20\, e^{0.0579236\, t}$

Here we use the given information about the fish population's growth and the solution to the differential equation to find the proportionality constant.

```
Solve[p1[14] == 45, k] // N

General::ivar : 0.05792358687259491` is not a valid variable. >>

Solve[True, 0.0579236]

k = 0.05792358687259491`

0.0579236
```

Now that we have found the proportionality constant we can model our differential equation using a slopefield. It gives the change of the fish popultion over time. We find that it increases or decreases to 1000. It is a stable equilibrium value. This is the fishes population model if the villagers do not remove any fish from the pond.

```
DEPlot[k * p * (1 - p/1000) - 10, {t, 0, 210}, {p, 10, 1300}, InitialPoints → {{0, 222}, {0, 223}, {0, 300}}]
```

DisplayTogether::obslt :

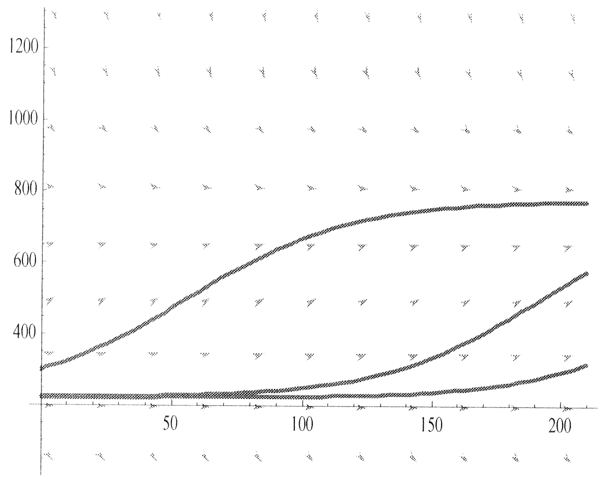
The DisplayTogether and DisplayTogetherArray functions are obsolete in Version 6. GraphicsArray and Show may be used directly in the same role.

Part::partw : Part 2 of {{{}, {}, {Hue[0.67, 0.6, 0.6], Thickness[0.005], Line[{{0., 222.}, «9», «68»}]}}} does not exist. »

Part::partw : Part 2 of {{{}, {}, {Hue[0.67, 0.6, 0.6], Thickness[0.005], Line[{{0., 222.}, «9», «68»}]}}} does not exist. »

Part::partw : Part 2 of {{{}, {}, {Hue[0.67, 0.6, 0.6], Thickness[0.005], Line[{{0., 223.}, «9», «68»}]}}} does not exist. »

General::stop : Further output of Part::partw will be suppressed during this calculation. »



Here we are trying to find the inital amount of fish needed to sustain the fish population if the villagers are fishing. At 222 the population is safe to be fished as long as the villagers no longer remove fish from the pond.

```
Solve[k * p * (1 - p/1000) - 10 == 0, p]
```

```
{{p → 221.866}, {p → 778.134}}
```

Below we enter values to see how many days it will take for the pond to gain enough fish. Once it has gained enough fish the villagers can start fishing agai and the army will no longer have to pay for their food.

```
Solve[p1[t] == 230, t]
```

Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. »

```
{{t → 42.165}}
```

To get the fish to a sustainable fishing equilibrium will take a couple hours over 42 days.

```
Solve[p1[t] == 778, t]
```

Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. »

```
{{t → 63.2039}}
```

To get the fish population to max. fishing equilibrium without any interference from the villagers will take a couple hours over 63 days.

P1(t) assumes exponential growth not logistic. Is that reasonable in the pond? Shouldn't you include carrying capacity.

Appendix C

- **Time to flush the pond of ACETYLENE DICHLORIDE (30 gallons)**
- **Analysis and Sensitivity Analysis with respect to cost.**

Here we develop a model that can tell us the amount of fish given any time and intial condition.

In[50]:= DSolve$\left[\left\{p'[t] == k * p[t] * \left(1 - \frac{p[t]}{1000}\right), p[0] == p0\right\}, p[t], t\right]$

Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. >>

Out[50]= $\left\{\left\{p[t] \to \frac{1.46424\times10^{17}\, 2.71828^{0.0579236\, t}}{1.46424\times10^{14}\, 2.71828^{0.0579236\, t} - 1.\left(-\frac{1.\,(1.46424\times10^{17} - 1.46424\times10^{14}\, p0)}{p0}\right)^{1.}}\right\}\right\}$

The below function represents the population of fish given an initial amount "p0", and a total time 't'. p0 is in fish and t is in days.

In[51]:= p[t_, p0_] = $\left(1.46423559259979*\text{^}17\; 2.71828182845904^{0.0579235868725949\, t}\right) \Big/ \left(1.46423559259*\text{^}14\; 2.7182818284590^{0.0579235868725949\, t} - 1.\left(-\frac{1.\,(1.46423559259979*\text{^}17 - 1.464235592599*\text{^}14\, p0)}{p0}\right)^{1.}\right)$

Out[51]= $\frac{1.46424\times10^{17}\, 2.71828^{0.0579236\, t}}{1.46424\times10^{14}\, 2.71828^{0.0579236\, t} - 1.\left(-\frac{1.\,(1.46424\times10^{17} - 1.46424\times10^{14}\, p0)}{p0}\right)^{1.}}$

Below we are trying to find a way to use our model to find minimum amount of days or fish need to reach the sustainable value of 230 fish in the pond. Once the population reaches 230 the villagers can start fishing again. The time to grow the fish and the amount of fish we initially put in the pond are the two factors affecting our total cost. The cost per fish is \$10. The cost per day the villagers have no food is \$100. Below is the cost of buying 230 fish from day one. The given time rounds to zero.

In[53]:= Solve[p[t, 230] == 230, t]

Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. >>

Out[53]= $\{\{t \to -3.16954\times10^{-11}\}\}$

This should not be necessary.

In[54]:= t1 = 0

Out[54]= 0

In[55]:= Cost1 = "\$" ((230 * 10) + (100 * t1))

Out[55]= 2300 \$

Appendix D

Next we see what the cost would be if we started with a small amount of fish, 10, and waited for it to grow to 230. We will save money on the amount of fish we purchase. Unfortunately, we will also be paying addtional for the villagers food for an addtional amout of time.

```
In[56]:= Solve[p[t, 10] == 230, t]

Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. >>

Out[56]= {{t → 58.4703}}

In[57]:= t2 = 59

Out[57]= 59

In[58]:= Cost2 = "$" ((59 * 100) + (10 * 10))

Out[58]= 6000 $
```

After conducting our cost analyis we find it is much cheaper to buy 230 fish from the hatchery and put them into the pond on the first day the pond is clea
Now that we have done this cost analysis we can add it to the cost of feeding the villagers while the pond is cleaned. Depending on how many gallons of chemical were spilt into the cost will determine the final cost. The range is from 3238 to 4836 dollars. We recommend cleaning out the pond as if there were 90 gallons in it, so there is no risk of the 230 new fish dieing.

```
In[59]:= TotalCost30G = Cost1 + "$" (100 * DaystoClean30G)

Out[59]= 3238.3 $

In[60]:= TotalCost90G = Cost1 + "$" (100 * DaystoClean90G)

Out[60]= 4836.29 $

In[61]:= TotalCost30G = Cost2 + "$" (100 * DaystoClean30G)

Out[61]= 6938.3 $

In[62]:= TotalCost90G = Cost2 + "$" (100 * DaystoClean90G)

Out[62]= 8536.29 $
```