Lesson 42 - Modeling with Differential Equations

Objectives

- Understand the following vocabulary:
  1. Independent vs. Dependent Variable
  2. Order of a Differential Equation
  3. Linearity of a Differential Equation
  4. Homogeneous or Non Homogeneous Differential Equation
  5. Analytic Solution, Graphical Solution, Numerical Solution
  6. General vs. Particular Solutions
  7. Equilibrium
- Classify a Differential Equation with respect to: Order, Linearity, and Homogeneity.
- Become familiar with differential equations that model exponential growth and the growth given some carrying capacity.
- Verify that a given function is a solution to a differential equation and/or initial value problem.

READ

- Stewart, Chapter 9.1, pages 567-570
- Supplemental Reading (Differential Equations), Course Guide

THINK ABOUT

- What is the meaning of discrete and continuous?
- What is the meaning of equilibrium?
- From the reading what does it mean for a function to solve a differential equation?

MATHEMATICA COMMANDS AND TASKS YOU NEED TO KNOW

Verifying a solution is done rather quickly in Mathematica. Follow the steps through this example:

Which of the following functions is a solution to the differential equation $y'' - y = 0$ (A) $y = \sin(x)$ (B) $y = 4e^{-x}$

1. Input the function that is given as a solution: `y[x_]:=Sin[x]`
2. Input the function as the differential equation dictates: `y''[x] - y[x] == 0` By using the `==` you are asking Mathematica to evaluate an expression. If the expression is correct you will see `True` as your output or an obviously true equation like $x = x$ or $e^x = e^x$. This means the function is a solution. Should you see anything else the function is not a solution or has been input incorrectly. Remember decimals in Mathematica are approximations. You should convert them to fractions for an exact number.
3. For solution (A) you should find that this is not a solution and for solution (B) you should find that this is a solution.

SUPPLEMENTAL READING DIFFERENTIAL EQUATIONS REVIEW

Calculus is the study of how we can express change mathematically: taking the ratio of the change in one quantity to the change in another yields an average rate of change, which in the limit becomes an instantaneous rate of change or derivative. So mathematically the investigation of change produces equations and expressions involving derivatives, Differential Equations.

In MA104, you learned how to find the derivative function $y' = \frac{dy}{dx} = f'(x)$ for the function $y = f(x)$. For example, given $f(x) = \sin(2x) + 3e^{-x}$, then $f'(x) = \frac{dy}{dx} = 2\cos(2x) - 3e^{-x}$. The result is a differential equation.

DIFFERENTIAL EQUATION-An equation relating an unknown function and any of its derivatives. If only one independent variable is present, the equation is called an Ordinary Differential Equation (ODE). If more than one independent variable is present, the equation is called a Partial Differential Equation (PDE).

1) $\dfrac{dy}{dx} + 2xy = e^x$ (ODE) 2) $\dfrac{\partial v}{\partial t} + \dfrac{\partial v}{\partial x} + 2v = 0$ (PDE)

From equation 1, we can recognize that y is a dependent variable and x is an independent variable. Can you think of ways to make that identification? In equation 2, we can conclude that v is a dependent variable. Notice that v depends on at least x and t. there might be even more independent variables.

In order to work with differential equations it is necessary to classify them. We will classify differential equations with respect to order, linearity, and homogeneity.

ORDER OF A DIFFERENTIAL EQUATION: The order of a differential equation is determined by the highest derivative in the equation. The differential equation:

$$\frac{dy}{dx} - 2y = 0$$

is said to be first order since it only contains a first order derivative while the differential equation:

$$y''' - 2y'' + 2y' + 2y = \cos(t)$$

is said to be third order since it contains a third order derivative.

LINEARITY OF A DIFFERENTIAL EQUATION: A differential equation is said to be linear if it contains no products, powers, or functions of the dependent variable or any of its derivatives. Linearity does not apply to the independent variable. The following differential equations are all linear:

$$x\frac{dy}{dx} + 2y = 2\sin(x) \qquad 2xy' - y = xe^{-x} \qquad (x^2+1)y' + xy = x$$

The following differential equation are all non-linear:

$$\frac{dy}{dx} + 2y^2 = 2\sin(y) \qquad y' - e^{-2y} = 0 \qquad (2x - y^2)y' + 2y = x$$

Notice in the non-linear examples there are products, powers, and functions of the dependent variable.

HOMOGENEITY OF A DIFFERENTIAL EQUATION: A differential equation is said to be homogeneous if all of the terms involve the dependent variable or one of its derivatives. If the independent variable is present by itself without the dependent variable included, it is non homogeneous. The following differential equations are all homogeneous:

$$\frac{dx}{dt} = 4xt \qquad \frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 2xy = 0$$

The following differential equations are nonhomogeneous can you see why?

$$\frac{dy}{dx} - y = 4 \qquad \frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 10y = e^x$$

We will focus on first and second order differential equations in this block.

SOLUTIONS OF DIFFERENTIAL EQUATIONS: One of the chief concerns about a differential equation is how to solve it. A solution to a differential equation is a function that satisfies the differential equation. There can be one, none, or infinitely many solutions. We will learn how to solve differential equations in three ways: analytically (exact), numerically (approximate), and graphically (approximate).

Analytic Solutions: We will learn various techniques to calculate exact or "closed form" solutions to differential equations. When the ability to get an analytic solution presents itself, you can verify that what you computed is a solution. The following example illustrates this process.

Example: Given the initial-value problem $\frac{dy}{dx} = y - x$ and the initial condition $y(0) = 2$, verify that $y = e^x + x + 1$ is a solution.

1. Take the appropriate amount of derivatives of the proposed solution:

$$y = e^x + x + 1 \quad \longrightarrow \quad \frac{dy}{dx} = e^x + 1$$

2. Substitute the derivatives into the differential equation:

$$\begin{aligned} \frac{dy}{dx} &= y - x \\ e^x + 1 &= e^x + x + 1 - x \\ e^x + 1 &= e^x + 1 \end{aligned} \tag{1}$$

   Which is true for all values of x. Therefore the function $y = e^x + x + 1$ satisfies the differential equation and is a "closed form" or analytic solution.

3. Check to see if the solution satisfies the initial condition:

$$y(x) = e^x + x + 1 \quad y(0) = e^0 + 0 + 1 = 2 \ \text{ so } \ y(0) = 2$$

   which satisfies the initial condition. Therefore the solution satisfies both the differential equation and the initial conditions.

Graphical Solutions: A solution to a differential equation is sometimes approximated by a graphical representation of a continuous function. A graphical solution to a first order differential equation is a curve whose slope at any point is the value of the derivative there, as given by the differential equation. Graphical solutions may be quantitative in nature; that is, the graph may be sufficiently precise so that the values of the solution function can be read directly from the graph. Or the solution may be qualitative, where the graph is imprecise as far as numerical values are concerned, yet still revealing of the general shape, features, and behavior of the solution curve. We will learn to create an interpret a graphical solution in the form of a slope field in lesson 38.

Numerical Solutions: A solution to a differential equation may also be approximated numerically. In this case, the form of the solution is a table of values of the dependent variable y for preselected values of the independent variable x. Numerical solutions are necessarily approximations to the true value of the solution and, as you might expect with any approximation method, we must be concerned with the accuracy of the numerical values. The numerical technique that we will learn in this course is Euler's Method, on lesson 39.