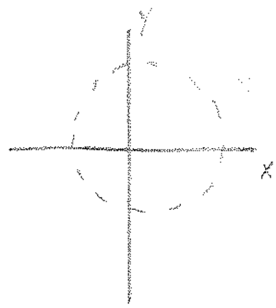
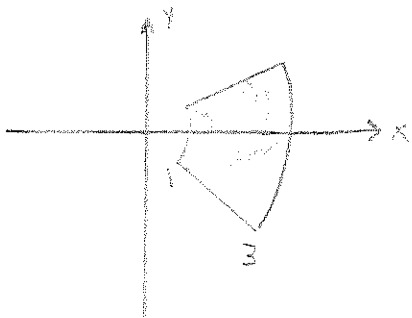
Mechanics Based Problems

1. Sketch the regions in the xy plane consisting of points whose polar coordinates satisfy the given conditions.

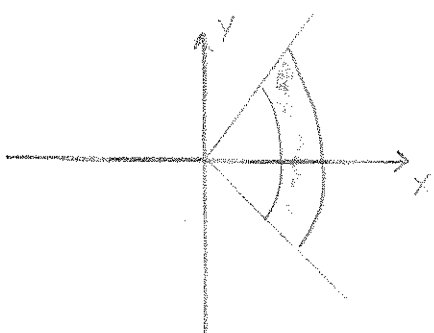
   (a) $r > 1$



   (b) $1 \leq r \leq 3$ and $-\pi/4 \leq \theta \leq \pi/4$



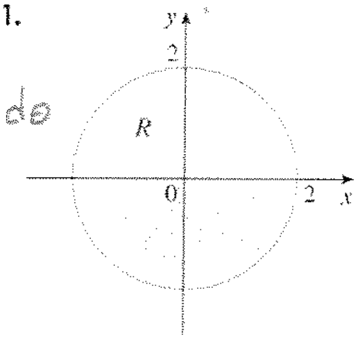
   (c) $2 < r < 3$ and $5\pi/3 \leq \theta \leq 7\pi/3$



2. A region R is shown below. Describe the regions by establishing the ranges for both r and θ if the region is better described by polar coordinates and x and y if the region is better described by cartesian coordinates. Then, establish an iterated integral over the region depicted using an arbitrary function $f(x, y)$. For example region 1 is a solid circle described as $0 \le r \le 2$ and $0 \le \theta \le 2\pi$ and $\int_0^{2\pi}\int_0^2 f(r\cos\theta, r\sin\theta)r\,dr\,d\theta$ would be the iterated integral of choice.

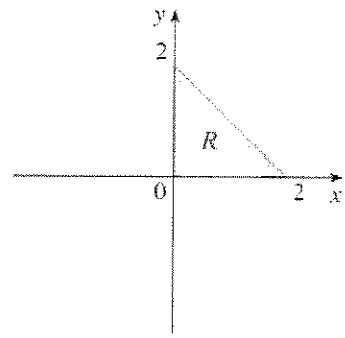
$0 \le r \le 2$
$0 \le \theta \le 2\pi$

1.



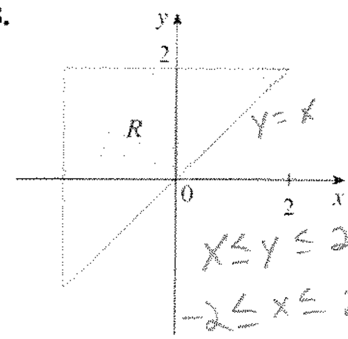
$\int_0^{2\pi}\int_0^2 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$

2.



$\int_0^2\int_0^{2-x} f(x,y)\,dy\,dx$

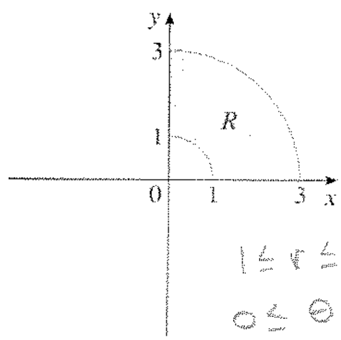
3.



$x \le y \le 2$
$-2 \le x \le 2$

$\int_{-2}^2\int_x^2 f(x,y)\,dy\,dx$

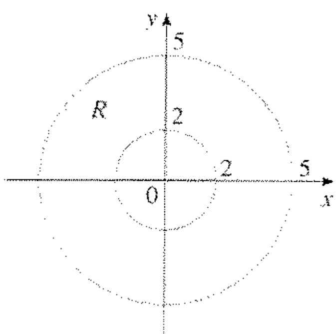
4.



$\int_0^{\pi/2}\int_1^3 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$

$1 \le r \le 3$
$0 \le \theta \le \pi/2$

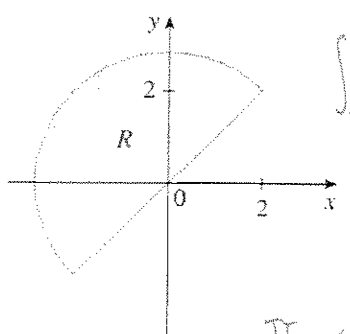
5.



$\int_0^{2\pi}\int_2^5 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$

$2 \le r \le 5$
$0 \le \theta \le 2\pi$

6.



$\int_{\pi/4}^{5\pi/4}\int_0^3 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$

$\pi/4 \le \theta \le 5\pi/4$
$0 \le r \le 3$

3. Sketch the region whose area is given by the integral and evaluate the integral:

$$\int_{\pi}^{2\pi} \int_{4}^{7} r\,dr\,d\theta$$

ANS

51.84

4. Evaluate the given integrals by changing to polar coordinates:

$$\int\int_{R} ye^{x}\,dA$$

where R is the region in the first quadrant enclosed by the circle $x^2 + y^2 = 25$

ANS

582.153

Problem Solving Problems

1. Use polar coordinates to find the volume of the solid under the paraboloid $z = x^2 + y^2$ and above the disk $x^2 + y^2 \leq 9$

18π ANS

2. A swimming pool is circular with a $40ft$ diameter. The depth is constant along east-west lines and increases linearly from $2ft$ at the south end to $7ft$ at the north end. Find the volume of water in the pool.

1800π ANS

3. A small yard sprinkler distributes water in a circular pattern of radius 15ft. It supplies water to a depth of $e^{-\sqrt{r}}$ feet per hour at a distance of r feet from the sprinkler.

   (a) What is the total amount of water supplied per hour to the region inside the circle of radius R ft centered at the sprinkler?

   (b) Determine an expression for the average amount of water per hour per square foot supplied to the region inside the circle of radius R ft.

   (c) Compute the average amount of water per hour per square foot supplied to the yard, based on the maximum range of the sprinkler.

$\underline{ANS}$

a) $\int_0^{2\pi} \int_0^{R} e^{-\sqrt{r}}\, r\, dr\, d\theta$

b) $\frac{1}{\pi R^2} \int_0^{2\pi} \int_0^{R} e^{-\sqrt{r}}\, r\, dr\, d\theta$

c) $\frac{1}{225\pi} \int_0^{2\pi} \int_0^{15} e^{-\sqrt{r}}\, r\, dr\, d\theta$

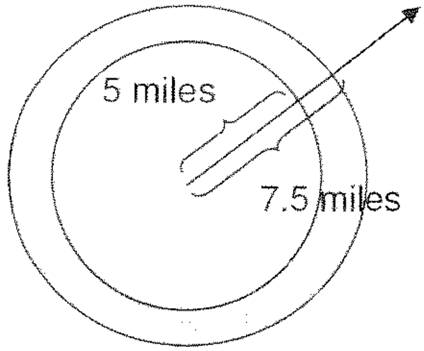
4. A lamina occupies the region inside the circle $x^2 + y^2 = 4$ but outside the circle $x^2 + y^2 = 1$ in the first quadrant. Find the center of mass if the density at any point is inversely proportional to its distance from the origin.

$\underline{ANS}$

$\bar{x} = 1.485$

$\bar{y} = 1.485$

5. The town of East Podunk is shaped like a series of two concentric circles. See the diagram provided. The population density in the inner city (a radius of 5 miles measured from the center of town) is determined to be $\rho(x, y) = 1200e^{-(x^2+y^2)}$. The population density in the outer region (out to a radius of 7.5 miles) has a constant population density of 50 people per square mile. What is the *total* population of East Podunk? What is the *average* population density of the town?



TOTAL Pop = 8679 people

Avg Pop = 49 people/mi²