The following differential equations are nonhomogeneous can you see why?

$$\frac{dy}{dx} - y = 4 \qquad \frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 10y = e^x$$

We will focus on first and second order differential equations in this block.

SOLUTIONS OF DIFFERENTIAL EQUATIONS: One of the chief concerns about a differential equation is how to solve it. A solution to a differential equation is a function that satisfies the differential equation. There can be one, none, or infinitely many solutions. We will learn how to solve differential equations in three ways: analytically (exact), numerically (approximate), and graphically (approximate).

Analytic Solutions: We will learn various techniques to calculate exact or "closed form" solutions to differential equations. When the ability to get an analytic solution presents itself, you can verify that what you computed is a solution. The following example illustrates this process.

Example: Given the initial-value problem $\frac{dy}{dx} = y - x$ and the initial condition $y(0) = 2$, verify that $y = e^x + x + 1$ is a solution.

1. Take the appropriate amount of derivatives of the proposed solution:

$$y = e^x + x + 1 \quad \longrightarrow \quad \frac{dy}{dx} = e^x + 1$$

2. Substitute the derivatives into the differential equation:

$$\begin{aligned} \frac{dy}{dx} &= y - x \\ e^x + 1 &= e^x + x + 1 - x \\ e^x + 1 &= e^x + 1 \end{aligned} \tag{1}$$

Which is true for all values of x. Therefore the function $y = e^x + x + 1$ satisfies the differential equation and is a "closed form" or analytic solution.

3. Check to see if the solution satisfies the initial condition:

$$y(x) = e^x + x + 1 \quad y(0) = e^0 + 0 + 1 = 2 \text{ so } y(0) = 2$$

which satisfies the initial condition. Therefore the solution satisfies both the differential equation and the initial conditions.

Graphical Solutions: A solution to a differential equation is sometimes approximated by a graphical representation of a continuous function. A graphical solution to a first order differential equation is a curve whose slope at any point is the value of the derivative there, as given by the differential equation. Graphical solutions may be quantitative in nature; that is, the graph may be sufficiently precise so that the values of the solution function can be read directly from the graph. Or the solution may be qualitative, where the graph is imprecise as far as numerical values are concerned, yet still revealing of the general shape, features, and behavior of the solution curve. We will learn to create an interpret a graphical solution in the form of a slope field in lesson 38.

Numerical Solutions: A solution to a differential equation may also be approximated numerically. In this case, the form of the solution is a table of values of the dependent variable y for preselected values of the independent variable x. Numerical solutions are necessarily approximations to the true value of the solution and, as you might expect with any approximation method, we must be concerned with the accuracy of the numerical values. The numerical technique that we will learn in this course is Euler's Method, on lesson 39.

Mechanics Based Problems

1. For the following problems state the order of the given ordinary differential equation. Determine whether the equation is linear or nonlinear, homogenous or non-homogenous.

   (a) $(1-x)y'' - 4xy' + 5y = \cos(x)$

   2nd order, linear, non-homogenous

   (b) $\frac{d^2u}{dr^2} + \frac{du}{dr} + u = \cos(r+u)$

   2nd order, nonlinear, homogeneous

   (c) $(\sin(\theta))y''' - (\cos(\theta))y' = 2$

   3rd order, linear, nonhomogeneous

2. For the following problems verify that the indicated function is a solution of the given differential equation.

   (a) $2y' + y = 0, \quad y = e^{-x/2}$

   $\boxed{\text{INSERT YOUR PROOF HERE}} \Rightarrow$ therefore, $y = e^{-x/2}$ is a solution to the DE

   (b) $y'' + y' - 6y = 0, \quad y_1 = c_1e^{2x}, \quad y_2 = c_1e^{2x} + c_2e^{-3x}$

   $\boxed{\text{INSERT YOUR PROOF HERE}} \Rightarrow$ therefore, y_1 and y_2 are both solutions to the DE

   (c) $xy' + y = 2x, \quad y = x - x^{-1}$

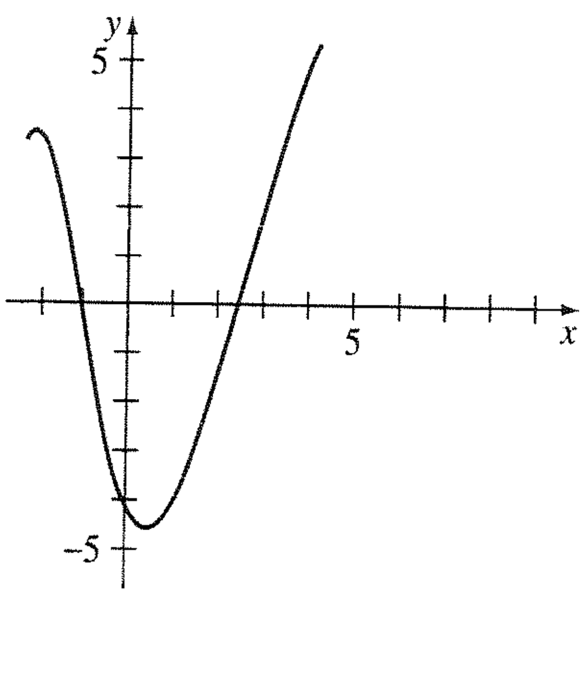
   $\boxed{\text{INSERT YOUR PROOF HERE}} \Rightarrow$ therefore, $y = x - x^{-1}$ is a solution to the DE

3. Given that $y = \frac{1}{x^2+c}$ is a solution of the first order DE $y' + 2xy^2 = 0$, find a solution of the first order IVP consisting of this differential equation and the initial condition $y(2) = 1/3$.

$$y = \frac{1}{x^2 - 1}$$

4. Given the following pairs of initial conditions and a graph of the particular solution to a second order differential equation of the form $d^2y/dx^2 = f(x, y, y')$ choose at least one set of initial conditions that agree with the solution curve.

   (a) $y(1) = 1$, $y'(1) = -2$
   (b) $y(-1) = 0$, $y'(-1) = -4$
   (c) $y(1) = 1$, $y'(1) = 2$
   (d) $y(0) = -1$, $y'(0) = 2$
   (e) $y(0) = -1$, $y'(0) = 0$
   (f) $y(0) = -4$, $y'(0) = -2$



Problem Solving Problems

1. Find values of r so that the function $y = e^{rx}$ is a solution of the given differential equation. Explain your reasoning.

   (a) $y'' + y' - 6y = 0$

   $r = -3, 2$

   (b) $y'' + 2y' + y = 0$

   $r = -1$

2. Find values of r so that the function $y = x^r$ is a solution of the differential equation,

$$xy'' + 2y' = 0$$

.

$r = 0, -1$

3. (a) Show that every member of the family of functions $y = Ce^{x^2/2}$ is a solution of the differential equation $y' = xy$.

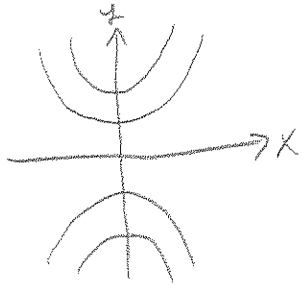
   (b) Illustrate part (a) by graphing several members of the family of solutions on a common screen.

   (c) Find a solution of the differential equation $y' = xy$ that satisfies the initial condition $y(0) = 5$.

   (d) Find a solution of the differential equation $y' = xy$ that satisfies the initial condition $y(1) = 2$.

a) INSERT YOUR PROOF HERE $\Longrightarrow$ therefore, for any value of C $y = Ce^{x^2/2}$ is a solution to $y' = xy$

b)



c) $y = 5e^{\frac{1}{2}x^2}$

d) $y = \frac{2}{e^{1/2}} e^{\frac{1}{2}x^2}$

4. A population of creatures is modeled by the differential equation

$$\frac{dP}{dt} = 1.2P\left(1 - \frac{P}{4200}\right)$$

(a) For what values of P is the population increasing?

$0 < P < 4,200$

(b) For what values of P is the population decreasing?

$P > 4,200$

(c) What are the equilibrium solutions?

$P = 0$

$P = 4200$

5. Psychologists interested in learning theory study **learning curves**. A learning curve is the graph of a function $P(t)$, the performance of someone learning a skill as a function of time t. The derivative dP/dt represents the rate at which performance improves.

   (a) When do you think P increases most rapidly? What happens to dP/dt as t increases? Explain.

   P increases most rapidly near $t=0$.

   $\frac{dP}{dt}$ decreases as t increases.

   PROVIDE YOUR EXPLANATION HERE

   (b) If M is the maximum level of performance of which the learner is capable, explain why the differential equation

   $$\frac{dP}{dt} = k(M-P), \quad k > 0$$

   is a reasonable model for learning.

   PROVIDE YOUR EXPLANATION HERE

   (c) Make a rough sketch of a possible solution of this differential equation.