Lesson 41 - Analytic Solutions I: Separation of Variables

Objectives

- Determine if a differential equation is separable.
- Determine a general solution to a differential equation using separation of variables.
- Determine a particular solution to a differential equation using separation of variables.

READ

- Stewart, Chapter 9.3, pages 580-585 (minus section on "Orthogonal Trajectories", pages 583-584).

THINK ABOUT

- What types of operations are usually not separable?
- What is the meaning of an antiderivative?

MATHEMATICA COMMANDS AND TASKS YOU NEED TO KNOW

Some differential equations can be extremely hard to solve by hand yet there are techniques to do it. We will focus on separation of variables as our first choice in solving equations but when it does not work we will use technology. Mathematica has a differential equations solving command called `DSolve`. Here is an example of how it works.

Given $dy/dx = x^2 + y^2$ and $y(0) = 5$ you can solve this unseparable equation with the command:

$$\texttt{DSolve}[\{y'[x] == x^2 + (y[x])^2, y[0] == 5\}, y[x], x]$$

Where $y(x)$ is your solution, x is your independent variable, y is your dependant variable.

You must use caution when using `DSolve` it is a very fragile command. It will find both general and particular solutions based on what you input into the command.

Mechanics Based Problems For problems 1-3 find solutions first without technology then use the DSolve command to check your work.

1. Find a general solution for $\frac{dy}{dx} = \sin(5x)$

$$y = -\frac{1}{5}\cos(5x) + C$$

2. Find a general solution for $\frac{dS}{dr} = kS$

$$S(r) = Ce^{kr}$$

3. Find a general solution for $\frac{dQ}{dt} = k(Q - 70)$

$$Q(t) = Ce^{kt} + 70$$

4. Find a particular solution for $\frac{dP}{dt} = \sqrt{Pt}, \quad P(1) = 2$

$$P(t) = \frac{1}{4}\left(\frac{2}{3}t^{3/2} + \sqrt{8} - \frac{2}{3}\right)^2$$

5. Find a general solution for $\frac{dN}{dt} + N = Ne^{t+2}$

$$N(t) = Ce^{(e^{t+2} - t)}$$

Problem Solving Problems

1. Freshly brewed coffee, with a temperature 95°C, is poured into a cup in a room with an ambient temperature of 20°C. If it is known that the coffee cools at a rate of 1°C per minute when its temperature 70°C, how long does it take for the coffee to cool to room temperature? Determine an analytical solution, then compare your most recent analytic analysis to both your graphical (Lesson 38) and your numerical (Lesson 39). Discuss the similarities or differences in your conclusion.

$$T(t) = 75e^{-0.02t} + 20$$

$$T(10) = 81.4°$$
$$T(20) = 70.27°$$
close to other estimates

2. A glucose solution is administered intravenously into the blood stream at a constant rate r. As the glucose is added, it is converted into other substances and removed from the blood stream at a rate that is proportional to the concentration at that time. Let $C(t)$ be the concentration of glucose in the blood stream at any time t and k is a positive proportionality constant.

   (a) Suppose that the initial concentration of glucose at time $t = 0$ is C_0. Determine the concentration at any time t.

$$C(t) = -\frac{1}{k}\left((r-kC_0)e^{-kt} - r\right)$$

   (b) Assuming that $C_0 < r/k$, find the limiting amount of glucose in the blood stream?

if $0 < C_0 < \frac{r}{k}$ then,

$$\frac{r}{k}\left(1-e^{-kt}\right) < C(t) < \frac{r}{k}$$

3. In a previous lesson, we formulated a model for learning in the form of the differential equation $\frac{dP}{dt} = k(M - P)$ where $P(t)$ measures the performance of someone learning a skill after training time, t, M is the maximum level of performance and k is a positive constant. Solve this differential equation to find an expression for $P(t)$. What is the limit of this expression?

$$\lim_{t \to \infty} P(t) = M$$

4. A thermometer reading 70° F is placed in an oven preheated to a constant temperature. Through a glass window in the oven door, an observer records that the thermometer reads 110° F after 30 seconds and 145° F after a minute. How hot is the oven?

$T_{OVEN} = 390°$ $(k \approx -0.267)$

5. A tank contains 200 liters of fluid in which 30 grams of salt is dissolved. Pure water is then pumped into the tank at a rate of 4 L/min; the well mixed solution is pumped out at the same rate. Find the number $A(t)$ of grams of salt in the tank for any time t.

$A(t) = 30e^{-\frac{1}{50}t}$