Lesson 46 - Spring Mass Systems I

Objectives

- Model initial value problems involving a spring mass system.
- Solve problems involving simple harmonic motion, free undamped motion, and damped motion.
- Understand critically damped, under damped, and over damped spring mass systems.
- Be able to describe the long term behavior of a spring mass system.

READ

- Stewart, Chapter 17.3, pages 1125-1128 (up to "Forced Vibrations")

THINK ABOUT

- What is the form for any linear second order differential equation, both homogeneous and non homogeneous?
- What is the form of any imaginary number?

MATHEMATICA COMMANDS AND TASKS YOU NEED TO KNOW

Some of the second order differential equations we will attempt to solve will not be easily solved by hand. We can use the command `DSolve` that we used earlier on these equations as well. Given a second order differential equation of the form:

$$ay'' + by' + cy = 0 \quad y(0) = y_0, y'(0) = y_1$$

The command to solve this in mathematica would be:

$$\texttt{DSolve}[\{ay''[t] + by'[t] + cy[t] == 0, y[0] == y_0, y'[0] == y_1\}, y[t], t]$$

Notice each equation has the == in it. Again, you must be very careful using the `DSolve` command. It is very sensitive to what you put into it.

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Problem Solving Problems

1. A mass weighing 24 pounds, attached to the end of a spring, stretches it 4 inches. Initially the mass is released from rest from a point 3 inches above the equilibrium position.

   (a) Develop an initial value problem that models the motion of the mass.

   $\frac{3}{4}x'' + 72x = 0, \quad x(0) = -\frac{1}{4}$

   $x'(0) = 0$

   (b) Determine the function that describes the position of the mass as a function of time.

   $x(t) = -\frac{1}{4}\cos(9.8t)$

   (c) When does the mass pass through the equilibrium position for the second time?

   $t = 0.48$ seconds

   (d) Identify and describe the type of motion this system represents.

   Undamped oscillatory motion about equilibrium (simple harmonic motion)

2. An 8 pound weight stretches a spring 4 feet. The spring-mass system resides in a medium offering a resistance to motion equivalent to 1.5 times the instantaneous velocity. The weight is released 2 feet above the equilibrium position with a downward velocity of 3 feet per second.

   (a) Develop an initial value problem that models the motion of the mass.

   $\frac{1}{4}x'' + 1.5x' + 2x = 0, \quad x(0) = -2, \quad x'(0) = 3$

   (b) Determine the function that describes the position of the mass as a function of time.

   $x(t) = \frac{1}{2}e^{-4t} - \frac{5}{2}e^{-2}$

   (c) When does the weight achieve the largest displacement from the equilibrium position?

   at time = 0 (release point)

   (d) Identify and describe the type of motion this system represents.

   Overdamped

3. A 4 foot spring measures 8 feet long after a mass weighing 8 pounds is attached to it. The medium through which the mass moves offers a damping force numerically equal to $\sqrt{2}$ times the instantaneous velocity. The mass is initially released from the equilibrium position with a downward velocity of 5 ft/sec.

   (a) Develop an initial value problem that models the motion of the mass.

   $$\frac{1}{4}x'' + \sqrt{2}x' + 2x = 0, \quad x(0) = 0, \quad x'(0) = 5$$

   (b) Determine the function that describes the position of the mass as a function of time.

   $$x(t) = 5te^{-2\sqrt{2}t}$$

   (c) When does the weight achieve the largest displacement from the equilibrium position? What is that distance?

   $$t = 0.3533$$
   $$x(0.3533) = 0.65$$

   (d) Identify and describe the type of motion this system represents.

   Critically Damped

4. A mass weighing 16 pounds stretches a spring 8/3 feet. The mass is initially released from rest from a point 2 feet below the equilibrium position, and the subsequent motion takes place in a medium that offers a damping force numerically equal to half the instantaneous velocity.

   (a) Develop an initial value problem that models the motion of the mass.

   $\frac{1}{2}x'' + \frac{1}{2}x' + 6x = 0, \quad x(0) = 2, \quad x'(0) = 0$

   (b) Determine the function that describes the position of the mass as a function of time.

   $x(t) = e^{-\frac{1}{2}t}\left(2\cos\left(\frac{\sqrt{47}}{2}t\right) + \frac{2}{\sqrt{47}}\sin\left(\frac{\sqrt{47}}{2}t\right)\right)$

   (c) When does the mass pass through the equilibrium position for the second time with an upwards velocity? (In otherwords, what is the period of this motion?)

   $t = 2.533$

   (d) Identify and describe the type of motion this system represents.

   Underdamped

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