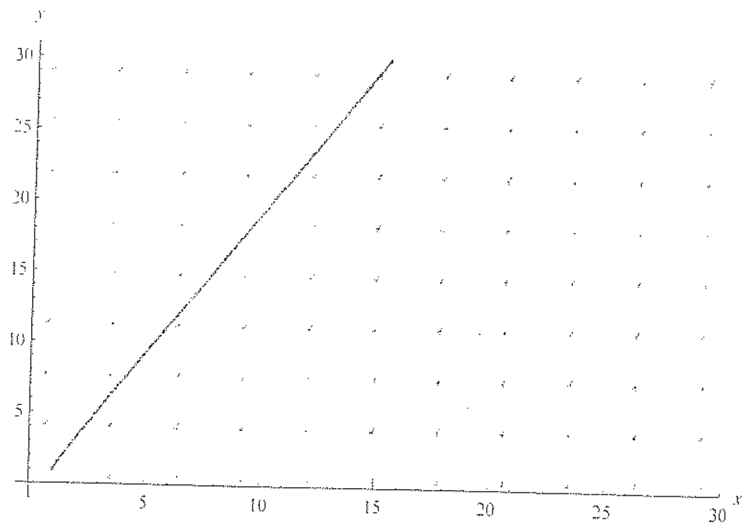
Problem Solving Problems

1. In the previous lessons, you wrote each of the following systems in matrix form, and then found the general solution to the system. For each of the systems, plot the phase portrait.

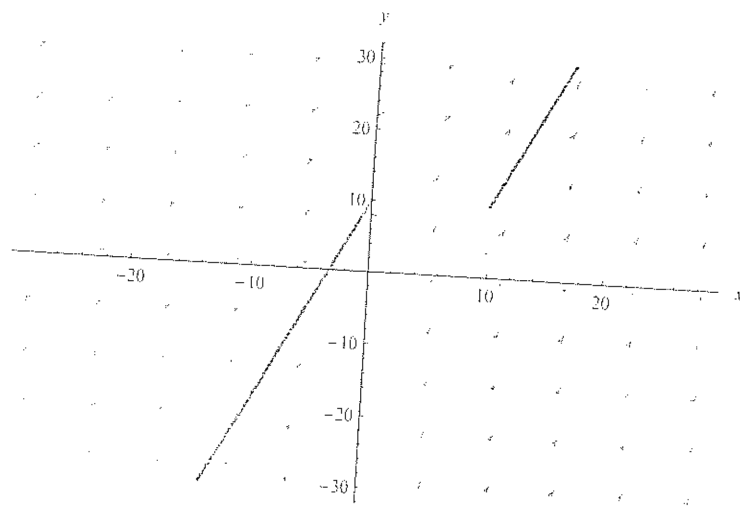
(a) $\dfrac{dx}{dt} = x + 2y$

$\dfrac{dy}{dt} = 4x + 3y$



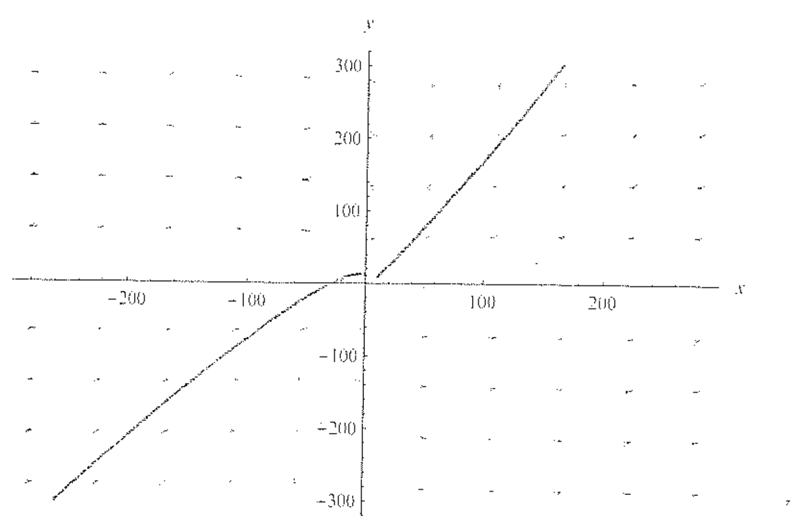
(b) $\dfrac{dx}{dt} = 3x - y$

$\dfrac{dy}{dt} = 9x - 3y$

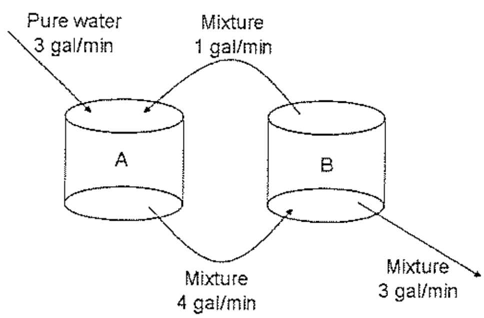


(c) $\dfrac{dx}{dt} = 6x - y$

$\dfrac{dy}{dt} = 5x + 2y$



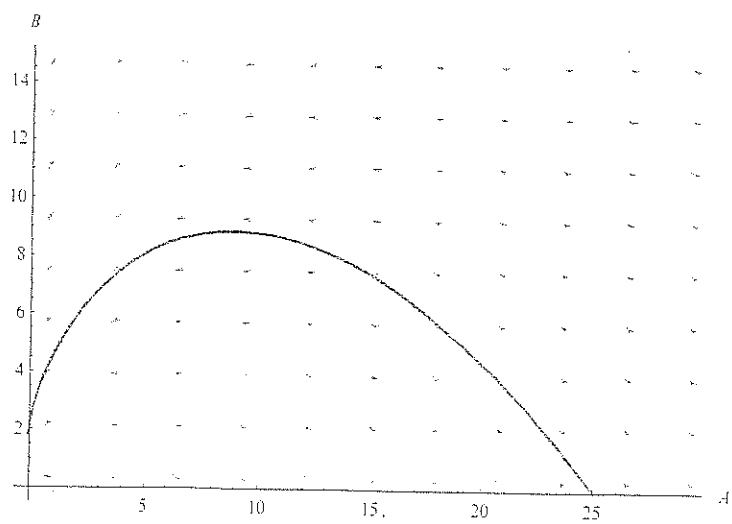
2. Recall the mixing problem from Lesson 49:
A tank contains 50 gallons of water in which 25 pounds of salt are dissolved. A second tank, B, contains 50 gallons of pure water. Liquid is pumped in and out of the tanks at rates shown in the figure below.



You have already developed the differential equations to model the system and have written the system in matrix form, and have found the general solution.

(a) Plot the phase portrait of the system, and plot the trajectory associated with the initial conditions.

(b) What happens to the amount of salt in the tank in the long run?



Salt goes to zero in long run

3. For the following system of differential equations, what happens in the long run if $x(0) = 100$, and $y(0) = 0$? What if $x(0) = -100$ and $y(0) = 0$?
$\dfrac{dx}{dt} = -x + 3y$, $\dfrac{dy}{dt} = -3x + 5y$

$X[0] = \begin{bmatrix} 100 \\ 0 \end{bmatrix} \Rightarrow$ x, y decrease

$X[0] = \begin{bmatrix} -100 \\ 0 \end{bmatrix} \Rightarrow$ x, y increase

4. Examine the phase portraits for the following systems, all of which have complex eigenvalues:

System 1:
$\frac{dx}{dt} = 4x + 5y$, $\frac{dy}{dt} = -2x + 6y$

System 2:
$\frac{dx}{dt} = 4x - 5y$, $\frac{dy}{dt} = 5x - 4y$

System 3:
$\frac{dx}{dt} = x - 8y$, $\frac{dy}{dt} = x - 3y$

Under what conditions will the phase portrait of a 2 x 2 homogeneous linear system with complex eigenvalues consist of...

(a) ...a family of closed curves?

complex repeat roots

(b) ...a family of spirals where the origin is a repeller?

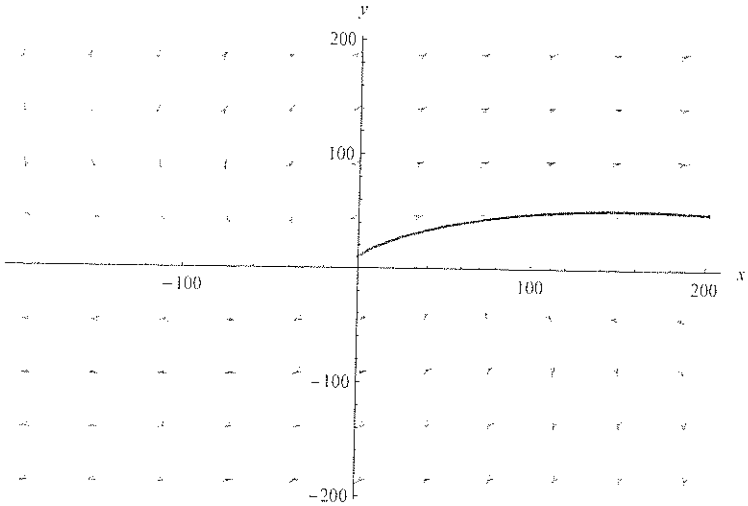
Complex roots, $\alpha > 0$

(c) ...a family of spirals where the origin is an attractor?

complex; $\alpha < 0$

System 1

```
PhasePlot[{4 x + 5 y, -2 x + 6 y}, {t, 0, 20}, {x, -200, 200}, {y, -200, 200},
  InitialPoints → {{0, 10}}, AxesLabel → {x, y}, AxesOrigin → {0, 0}]
```

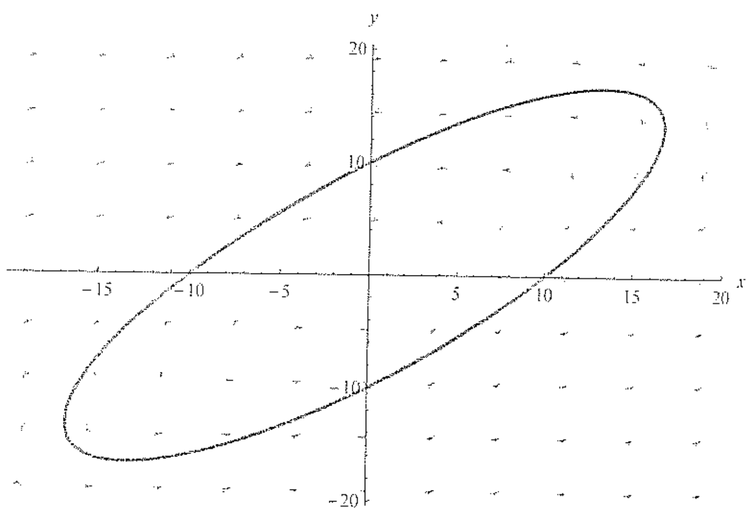



```
Eigensystem[( 4  5
             -2  6 )]
```

$$\left\{\{5+3\,i,\ 5-3\,i\},\ \left\{\left\{\frac{1}{2}-\frac{3\,i}{2},\ 1\right\},\ \left\{\frac{1}{2}+\frac{3\,i}{2},\ 1\right\}\right\}\right\}$$

System 2

```
PhasePlot[{4 x - 5 y, 5 x - 4 y}, {t, 0, 20}, {x, -20, 20}, {y, -20, 20},
  InitialPoints → {{10, 0}}, AxesLabel → {x, y}, AxesOrigin → {0, 0}]
```

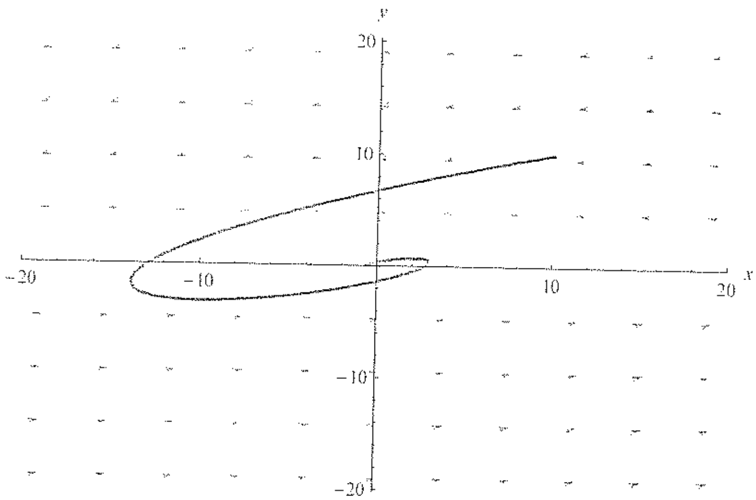



```
Eigensystem[( 4  -5
              5  -4 )]
```

$$\left\{\{3\,i,\ -3\,i\},\ \left\{\left\{\frac{4}{5}+\frac{3\,i}{5},\ 1\right\},\ \left\{\frac{4}{5}-\frac{3\,i}{5},\ 1\right\}\right\}\right\}$$

System 3

```
PhasePlot[{x - 8 y, x - 3 y}, {t, 0, 20}, {x, -20, 20}, {y, -20, 20},
  InitialPoints → {{10, 10}}, AxesLabel → {x, y}, AxesOrigin → {0, 0}]
```




```
Eigensystem[(1  -8
             1  -3)]
```

$\{\{-1+2i, -1-2i\}, \{\{2+2i, 1\}, \{2-2i, 1\}\}\}$