MA255 09-02
Mathematical Modeling & Introduction to Differential Equations
Course Overview and Objectives (as of 07JAN09)

Main Text: Differential Equations: An Introduction to Modern Methods & Applications, Brannan & Boyce, 2007.
Supplementary Text: Differential Equations with Mathematica; Hunt, Lipsman, Osborn, Outing, & Rosenberg; 2008.

Introduction

This is the second course of a two-semester advanced mathematics sequence for selected cadets who have validated single variable calculus and demonstrated strength in the mathematical sciences. It is designed to provide a foundation for the continued study of mathematics, sciences, and engineering. This course emphasizes the interaction between mathematics and the physical sciences through modeling with differential equations. Topics include a study of first order differential equations, first order difference equations, second order linear equations, systems of first order linear equations, Laplace transforms, numerical methods, and nonlinear equations and stability. An understanding of course material is enhanced through the use of *Mathematica* (computer algebra system) and *Excel* (spreadsheet).

Students are expected to become proficient in the following block-specific objectives.

Block 1 (Chapters 1 & 2)

First Order Differential Equations

This block aims to introduce the concept of differential equations, particularly first order differential equations, within a modeling framework. Students will create mathematical models with differential equations, conduct qualitative analysis, learn techniques to solve linear differential equations, and gain some understanding of the history of this important branch of mathematics. Students will build on their existing *Mathematica* & *Excel* knowledge by finding exact solutions to first order differential equations, as well as use numerical methods to approximate solutions.

Chapter 1 – Introduction

a. Derive differential equations that mathematically model simple problems (1.1).
b. Graph a direction field for a first order ODE and sketch approximate solutions (1.1).
c. Graph the integral curves of a general solution (1.2).
d. Understand what an initial value problem is, and how to show a given function is a solution to one (1.2).
e. Understand when and how to apply Euler's method to solve first order initial value problems (1.3).
f. Use a chart to compare the exact solution with Euler's method for several different step sizes (1.3).

g. Understand the difference between an ordinary differential equation and a partial differential equation (1.4).

h. Classify differential equations with respect to order and linearity (1.4).

Chapter 2 – First Order Differential Equations

a. Compute the integrating factor and general solution for a linear differential equation (2.1).

b. Determine the general solution for a separable equation (2.2).

c. Understand the three identifiable steps of the mathematical modeling process (2.3).

d. Identify differences between linear and nonlinear differential equations (2.4).

e. Recognize autonomous equations and utilize the direction field to represent solutions to them (2.5).

f. Identify the logistic equation, determine the equilibrium solutions, and classify them as asymptotically stable, semi-stable, or unstable (2.5).

g. Understand the definition of the threshold level (2.5).

h. Determine if a differential equation is exact (2.6).

i. Use integrating factors to convert a differential equation that is non exact into an exact equation (2.6).

j. Determine the general solution for an exact equation (2.6).

k. Understand the differences between global truncation error, local truncation error, and round-off error (2.7).

l. Define bounds on the local truncation error and use this information to adjust the step size in order to obtain an accurate approximation (2.7).

m. Understand when and how to apply the improved Euler method to solve first order initial value problems (2.8).

n. Understand when and how to apply the Runge-Kutta method to solve first order initial value problems (2.8).

Block 2 (Chapters 3 & 4)

Systems of Two First Order Equations & Second Order Linear Equations

This block consists of two parts, systems of two first order equations and second order linear equations. Chapter 3 will further expand the students' modeling repertoire by introducing systems of two differential equations and the associated linear algebra properties inherent in these systems. While Chapter 4 seeks to build on students' ability to solve second order linear differential equations. Students will use second order linear equations with constant coefficients to model a variety of mechanical and electrical vibration problems. Students will continue to use technology to graph solutions and use numerical methods to approximate solutions.

Chapter 3 – Systems of Two First Order Equations

a. Understand and apply the properties of matrices (A.1).

b. Find the solution to a set of linear algebraic equations (3.1).

c. Find the eigenvalues and eigenvectors of a matrix (3.1).

d. Determine the phase plane and phase portraits of a 2×2 linear system (3.2).

e. Classify the critical point of a system of two linear ODEs with respect to type and stability (3.2).

f. Plot the direction field for a system of two linear ODEs (3.2).

g. Determine whether a system of linear ODEs is homogenous or nonhomogeneous (3.2).

h. Understand the components of a solution to a system of two linear ODEs (3.2).

i. Determine equilibrium solutions for an autonomous system of two linear ODEs (3.2).

j. Transform a second order differential equation into a system of first order equations (3.2).

k. Understand the definition of a fundamental set of solutions (3.3).

l. Use the Wronskian to determine if a set of solutions forms a fundamental set for a system of two linear ODEs (3.3).

m. Determine the stability of systems of two linear ODEs, and be able to classify a node as a nodal sink, nodal source, or saddle point (3.3).

n. Find the general solution of a system of two linear ODEs with real and different eigenvalues (3.3).

o. Find the general solution of a system of two linear ODEs with complex eigenvalues (3.4).

p. Find the general solution of a system of two linear ODEs with repeated eigenvalues (3.5).

q. Understand stability as related to nonlinear systems (3.6).

r. Without the use of a computer, sketch the phase portrait of a nonlinear system (3.6).

s. Understand when and how to apply the Euler, improved Euler, and Runge-Kutta methods to approximate solutions of initial value problems for a system of first order equations (3.7).

Chapter 4 – Second Order Linear Equations

a. Understand the components of an initial value problem for a second order linear equation (4.1).

b. Determine whether an equation is homogenous or nonhomogeneous (4.1).

c. Determine whether an equation is linear or nonlinear (4.1)

d. Understand the implications of the Existence and Uniqueness Theorem and the Principle of Superposition (4.2).

e. Use the Wronskian to determine fundamental solution sets to homogenous equations (4.2)

f. Understand when functions are considered linearly dependent (4.2).

g. Understand the relationship between a fundamental set of solutions, linear independence, and a non-zero Wronskian (4.2).

h. Use the real, unique roots of the characteristic equation to find the general solution of a homogenous equation with constant coefficients (4.3).

i. Use the initial condition to find the specific solution of a second order differential equation (4.3).

j. Use the real, repeated roots of the characteristic equation to find the general solution of a homogenous equation with constant coefficients (4.3).

k. Use the complex valued roots of the characteristic equation to find the general solution of a homogenous equation with constant coefficients (4.4).

l. Determine the period, natural frequency, amplitude, and phase of undamped and unforced vibrations (4.5).

m. Determine the quasi frequency and quasi period of damped vibrations (4.5).

n. Understand the three different kinds of dampening: overdamped, underdamped, and critically damped (4.5).

o. Understand that solutions to second order nonhomogenous equations have two components – the complementary solution and the particular solution (4.6).

p. Use the method of undetermined coefficients to solve for the particular solution of a nonhomogenous equation (4.6).

q. Understand the difference between forced and unforced vibrations (4.7).

r. Know the difference between the transient solution and the steady-state solution (4.7).

s. Understand the difference between beats and resonance, and when they occur (4.7).

Block 3 (Chapters 5 – 7)

Laplace Transform, Systems of First Order Linear Equations, & Nonlinear Differential Equations

This block consists of three parts, Laplace transforms, a general treatment of systems of first order linear equations, and nonlinear differential equations. Chapter 5 will introduce the Laplace transform method in solving linear differential equations, emphasizing problems typical in engineering applications. Chapter 6 will build on elementary theory and solution techniques for first order linear systems introduced to students in a two dimensional setting in Chapters 3 & 4. Chapter 7 invites students to investigate systems of nonlinear equations using analytical, numerical, and qualitative techniques.

Chapter 5 – The Laplace Transform

a. Understand the definition of the Laplace transform (5.1).

b. Understand the properties of the Laplace transform (5.2).

c. Determine the inverse Laplace transform of a function using a table of elementary Laplace Transforms (5.3).

d. Determine the solution to an initial value problem for a second order linear differential equation using Laplace transforms (5.4).

e. Determine the solution to an initial value problem for a system of two linear equations using Laplace transforms (5.4).

f. Understand the properties of discontinuous & periodic functions (5.5).

g. Find the Laplace transform of discontinuous & periodic functions (5.5).

h. Determine the solution to an initial value problem for a second order linear differential equation with a discontinuous forcing function using Laplace transforms (5.6).

Chapter 6 – Systems of First Order Linear Equations

a. Transform an n^{th} order differential equation into a system of first order equations (6.1).

b. Verify solutions to matrix differential equations (6.1).

c. Understand the definition of a fundamental set of solutions (6.2).

d. Find the general solution of a system of linear ODEs with real and different eigenvalues (6.3).

e. Find the general solution of a system of linear ODEs with complex eigenvalues (6.4).

f. Understand the properties of the fundamental matrix and be able to determine what it is for a given system of first order linear equations (6.5).

g. Understand the properties of the matrix exponential function and be able to determine what it is for a given system of first order linear equations (6.5).

h. Determine the solution to an initial value problem for a system of linear equations using the fundamental matrix of the system (6.5).

i. Find the general solution of a nonhomogeneous linear system (6.6).

Chapter 7 – Nonlinear Differential Equations and Stability

a. Classify the critical point of a 2×2 autonomous system with respect to type and stability (7.1).

b. Become familiar with the separatrix and the basin of attraction (7.1).

c. Determine the trajectories for a system of ODEs (7.1).

d. Determine whether a system of ODEs is almost linear (7.2).

e. Classify the critical point of an almost linear system with respect to type and stability (7.2).

f. Plot the direction field and phase portraits for competing species problems (7.3).

g. Plot the direction field and phase portraits for predator-prey problems (7.4).