Day 1 Lets students develop from Laws 1-3 w/ some slight notation

Day 2/3 this paper

11.7 THE ONE-DIMENSIONAL HEAT EQUATION

In the investigation of the flow of heat in a conducting body, the following three laws have been deduced from experimentation.

LAW 1 Heat will flow from a region of higher temperature to a region of lower temperature.

LAW 2 The amount of heat in the body is proportional to the temperature of the body and to the mass of the body.

LAW 3 Heat flows across an area at a rate proportional to the area and to the temperature gradient (that is, the rate of change of temperature with respect to distance where the distance is taken perpendicular to the area).

We consider a rod of length l and constant cross sectional area A. The rod is assumed to be made of material that conducts heat uniformly. The lateral surface of the rod is insulated so that the streamlines of heat are straight lines perpendicular to the cross sectional area A. The x-axis is taken parallel to and in the same direction as the flow of heat. The point $x = 0$ is at one end of the rod and the point $x = l$ is at the other end. ρ denotes the density of the material, and c (a constant) denotes the *specific heat* of the material. (Specific heat is the amount of heat energy required to raise a unit mass of the material one unit of temperature change.) $u(x, t)$ denotes the temperature at time t (> 0) in a cross sectional area A, x units from the end $x = 0$. Consider a small portion of the rod of thickness Δx that is between x and $x + \Delta x$. The amount of heat in this portion is, by Law 2, $c\rho A\Delta x u$. Thus, the time rate of change of this quantity of heat is $c\rho A\Delta x \frac{\partial u}{\partial t}$. Thus,

$$c\rho A\Delta x \frac{\partial u}{\partial t} = (\text{rate into this portion}) - (\text{rate out of this portion}).$$

From Law 3 we have

$$\text{rate in} = -kA\frac{\partial u}{\partial x}\bigg|_{x},$$

$$\text{rate out} = -kA\frac{\partial u}{\partial x}\bigg|_{x+\Delta x},$$

where the minus sign is a consequence of Law (1) and our assumption regarding the orientation of the x-axis. The constant of proportionality k is known as the *thermal conductivity*. Thus,

$$c\rho A\Delta x \frac{\partial u}{\partial t} = -kA\frac{\partial u}{\partial x}\bigg|_{x} + kA\frac{\partial u}{\partial x}\bigg|_{x+\Delta x}$$

We offer a copy of the material on the development of the heat equation model from the book, *An Introduction to Differential Equation with Difference Equations, Fourier Series, and Partial Differential Equations*, by N. Finizio and G. Ladas, Belmont CA: Wadsworth Pub. Co. 1982. pp. 455-456.

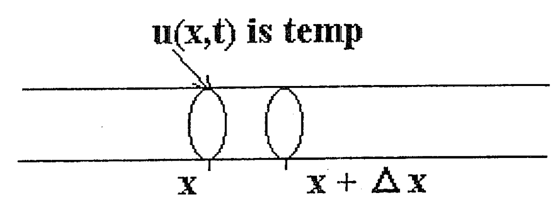
or

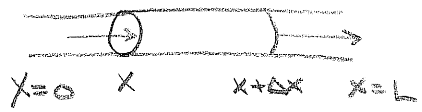
$$\frac{\partial u}{\partial t} = \frac{k}{c\rho}\left[\frac{\left.\frac{\partial u}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial u}{\partial x}\right|_{x}}{\Delta x}\right].$$

Taking the limit as Δx tends to zero, we obtain the *one-dimensional heat equation*

$$u_t - au_{xx} = 0, \tag{1}$$

where $a = k/c\rho$ is known as the *diffusivity*.





$u(x,t)$ = temp °C in rod @ x cm & t sec

- assumptions
  1 - heat flow from higher to lower temp
  2 - amt of heat in the body is proportional to the temp of the body & to the mass of the body
  3 - heat flows across an area @ a rate proportional to the area & to the temp. gradient.
- before we see how it changes, lets find out how much is in there -

want calories

setting up the model

$$\frac{cal}{g\,°C} \cdot \frac{g}{cm^3} \cdot cm^2 \cdot cm \cdot °C = \text{calories} \quad ✓$$

$$c \cdot \rho \cdot A \cdot \Delta x \cdot u(x,t)$$

variables
c = specific heat
ρ = density
A = cross sectional area
Δx = length
$u(x,t)$ = temp

amount of heat energy = $c \rho A \Delta x \, u(x,t)$

Δ heat energy w.r.t. time = $\frac{\partial}{\partial t} c \rho A \Delta x \, u$

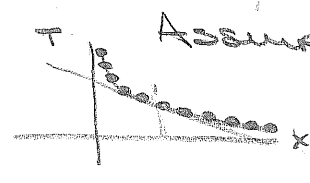
$$= c \rho A \Delta x \frac{\partial u}{\partial t}$$

Δ heat energy w.r.t. posn = (rate into portion - rate out portion

Assumption 3 $\Rightarrow$ rate in = $-kA \frac{\partial u}{\partial x}\Big|_x$

rate out = $kA \frac{\partial u}{\partial x}\Big|_{x+\Delta x}$

Assumption 1 for direction



thermal conductivity

so,

Δ heat energy w.r.t time = Δ heat energy w.r.t position

$$C \rho A \Delta x \frac{\partial u}{\partial t} = -kA \frac{\partial u}{\partial x}\Big|_x + kA \frac{\partial u}{\partial x}\Big|_{x+\Delta x}$$

so...

$$\Delta x \frac{\partial u}{\partial t} = \frac{-k}{C\rho} \frac{\partial u}{\partial x}\Big|_x + \frac{k}{C\rho} \frac{\partial u}{\partial x}\Big|_{x+\Delta x}$$

$$\Delta x \frac{\partial u}{\partial t} = \frac{k}{C\rho}\left(\frac{\partial u}{\partial x}\Big|_{x+\Delta x} - \frac{\partial u}{\partial x}\Big|_x \right)$$

$$\frac{\partial u}{\partial t} = K \frac{\left(\frac{\partial u}{\partial x}\Big|_{x+\Delta x} - \frac{\partial u}{\partial x}\Big|_x \right)}{\Delta x}$$ where $K = \frac{k}{C\rho}$

where $k = \frac{cal}{cm^3\ ^\circ C}$

take $\lim\limits_{\Delta x \to 0} \left[\frac{\partial u}{\partial t} = K \frac{\left(\frac{\partial u}{\partial x}\Big|_{x+\Delta x} - \frac{\partial u}{\partial x}\Big|_x \right)}{\Delta x} \right]$

now observing heat @ a point

$$\frac{\partial u}{\partial t}(x,t) = K \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}(x,t) \right)$$

$$\frac{\partial u}{\partial t}(x,t) = K \frac{\partial^2 u}{\partial x^2}(x,t)$$

$$u_t = K\, u_{xx}$$

diffusivity