SOLUTIONS

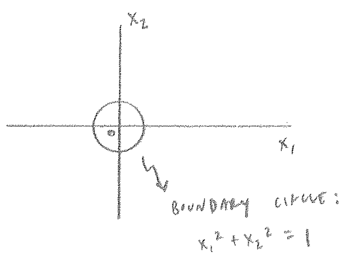
MA364 Lesson 18: Applications of Eigenvalues/Eigenvectors
MAJ Sullivan

1. (Stretching of an Elastic Membrane). An elastic membrane in the x_1x_2 – plane with boundary circle $x_1^2 + x_2^2 = 1$ is stretched so that a point $P:(x_1, x_2)$ is mapped to the point $Q:(y_1, y_2)$ according to the linear transformation:

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{\vec{y}} = \underbrace{\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} \Rightarrow \begin{aligned} y_1 &= 5x_1 + 3x_2 \\ y_2 &= 3x_1 + 5x_2 \end{aligned}$$

NOTE !

Find the principal directions, i.e. the directions of the position vector OP for which the direction of the position vector OQ is the same or exactly opposite. What shape does the boundary circle take under the deformation?



$P:(x_1, x_2)$, $Q:(y_1, y_2)$

LET $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{OP}$ AND $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \vec{OQ}$

$\Rightarrow$ WE ARE LOOKING FOR VECTORS $\vec{x}$ SUCH THAT

$$\underbrace{\vec{y} = A\vec{x} = \lambda\vec{x}}_{\text{THIS IS AN EIGENVALUE PROBLEM!}}$$

NOTE: WE DON'T ASSUME $\lambda = I$ FOR PRINCIPAL DIRECTION, BECAUSE WE ARE CONCERNED W/ DIRECTION, NOT MAGNITUDE

$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix} = (5-\lambda)^2 - 9 = 0$

$25 - 10\lambda + \lambda^2 - 9 = 0 \Rightarrow \lambda^2 - 10\lambda + 16 = (\lambda - 8)(\lambda - 2) = 0$

$\Rightarrow \lambda = 8, 2$

Find the corresponding eigenvectors:

$\boxed{\lambda = 8}$ $(A - \lambda I)\vec{x} = \vec{0} \Rightarrow \begin{bmatrix} 5-8 & 3 \\ 3 & 5-8 \end{bmatrix} \vec{x} = \vec{0}$

$\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} R_2 + R_1 \Rightarrow \begin{bmatrix} -3 & 3 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{aligned} -3x_1 + 3x_2 &= 0 \\ 0 &= 0 \end{aligned}$

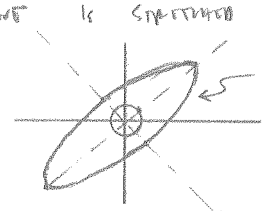
$\therefore x_1 = x_2$

$\Rightarrow$ AN EIGENVECTOR FOR $\lambda = 8$ IS $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\boxed{\lambda = 2}$ IT CAN BE SHOWN SIMILARLY THAT AN EIGENVECTOR FOR $\lambda = 2$ IS $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

THESE TWO EIGENVECTORS MAKE $\frac{\pi}{4}$ AND $\frac{3\pi}{4}$ $\measuredangle$'s W/ THE $+x_1$-DIRECTION, RESPECTIVELY

$\Rightarrow$ BY THE SET-UP OF OUR PROBLEM, THESE ARE THE PRINCIPAL DIRECTIONS AND OUR EIGENVALUES SHOW THAT THE MEMBRANE IS STRETCHED BY A FACTOR OF 8 AND 2 IN THEIR CORRESPONDING PRINCIPAL DIRECTIONS

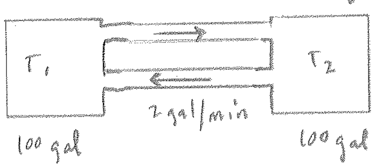


DEFORMED MEMBRANE

$(1,1)$ MAPS TO $(8,8)$, $(1,-1) \to (2,-2)$, $(-1,-1) \to (-8,-8)$, $(-1,1) \to (-2,2)$ ETC.

2. (System of ODEs – Mixing Problem). Tanks T_1 and T_2 each initially contain 100 gallons of water. At time $t = 0$, 150 pounds of fertilizer is dissolved in T_2. Liquid circulates between the two tanks at a rate of 2 gallons per minute. Find the amount of fertilizer in the tanks at any time.

150 lb fertilizer @ $t=0$



LET y_1 = AMOUNT OF FERTILIZER (in lbs) IN T_1 @ TIME t (min)

y_2 = " " " " " T_2 " " "

$\frac{\text{lb}}{\text{min}}$

$$\frac{dy_1}{dt} = \text{rate in} - \text{rate out} = 2\,\frac{\text{gal}}{\text{min}}\left(\frac{y_2}{100}\,\frac{\text{lb}}{\text{gal}}\right) - 2\,\frac{\text{gal}}{\text{min}}\left(\frac{y_1}{100}\,\frac{\text{lb}}{\text{gal}}\right)$$

$$= .02\,y_2 - .02\,y_1$$

$$\boxed{\text{IC's: } y_1(0) = 0 \quad y_2(0) = 150}$$

$$\frac{dy_2}{dt} = .02\,y_1 - .02\,y_2$$

$$\Rightarrow \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -.02 & .02 \\ .02 & -.02 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \Rightarrow \vec{y}\,' = A\vec{y}$$

OUR "GUESS" AS TO THE SOLUTION OF THIS VECTOR DIFFERENTIAL EQUATION IS $\vec{y} = \vec{x}\,e^{\lambda t}$

$$\Rightarrow \underline{\vec{y}\,' = \lambda\vec{x}\,e^{\lambda t}} = A\vec{y} = \underline{A\vec{x}\,e^{\lambda t}}$$

$$\lambda\vec{x}\,\cancel{e^{\lambda t}} = A\vec{x}\,\cancel{e^{\lambda t}} \Rightarrow A\vec{x} = \lambda\vec{x} \quad \text{AN EIGENVALUE PROBLEM!}$$

IT CAN BE SHOWN THAT $\lambda = 0, -.04$ w/ CORRESPONDING EIGENVECTORS $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ AND $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

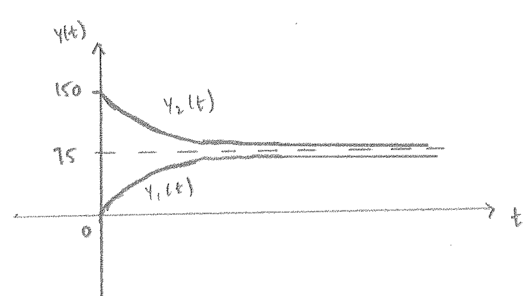
this is fine! only eigenvectors need to be $\neq 0$

THE GENERAL SOLUTION TO OUR VECTOR DIFFERENTIAL EQUATION IS ANY LINEAR COMBINATION OF A BASIS OF SOLUTIONS

$$\Rightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}\cancel{e^{0t}}^{\,1} + c_2\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{-0.04t} \Rightarrow \begin{aligned} y_1 &= c_1 + c_2\,e^{-0.04t} \\ y_2 &= c_1 - c_2\,e^{-0.04t} \end{aligned}$$

APPLYING THE INITIAL CONDITIONS $y_1(0) = 0$ AND $y_2(0) = 150$ YIELDS $c_1 = 75$, $c_2 = -75$

$$\therefore \quad y_1(t) = 75 - 75\,e^{-0.04t}$$
$$y_2(t) = 75 + 75\,e^{-0.04t}$$



ANS

3. (Markov Processes). Suppose that in 2004 the state of land use in a city of 60 square miles of built-up area is C: Commercially Used 25%, I: Industrially Used 20%, and R: Residentially Used 55%. Find the limit states assuming that the transition probabilities for 5-year intervals are given by the matrix A and remain practically the same over the time considered.

$$A = \begin{array}{c} \begin{array}{ccc} \textit{From } C & \textit{From } I & \textit{From } R \end{array} \\ \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.1 & 0.9 & 0 \\ 0 & 0.2 & 0.8 \end{bmatrix} \end{array} \begin{array}{c} \\ \textit{To } C \\ \textit{To } I \\ \textit{To } R \end{array}$$

A is a stochastic matrix → a square matrix w/ all elements non-negative and all column sums equal to 1

This example concerns a Markov process, a process for which the probability of entering a certain state depends only on the last state occupied

NOTE THAT $A^T = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.1 & 0.9 & 0 \\ 0 & 0.2 & 0.8 \end{bmatrix}$ AND SINCE ALL THE COLUMN SUMS OF A ARE EQUAL TO 1 $\Rightarrow A^T \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = (1) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

THUS, $\lambda = 1$ IS AN EIGENVALUE OF A^T AND BY THM #3, § 8.1, IT IS ALSO AN EIGENVALUE OF A

$$\Rightarrow (A - \lambda I)\vec{x} = \vec{0} \Rightarrow \begin{bmatrix} 0.7-1 & 0.1 & 0 \\ 0.2 & 0.9-1 & 0.2 \\ 0.1 & 0 & 0.8-1 \end{bmatrix} \vec{x} = \vec{0}$$

$$\begin{bmatrix} -0.3 & 0.1 & 0 \\ 0.2 & -0.1 & 0.2 \\ 0.1 & 0 & -0.2 \end{bmatrix} \begin{array}{l} \\ 1.5 R_2 + R_1 \\ 3 R_3 + R_1 \end{array}$$

$$\begin{bmatrix} -0.3 & 0.1 & 0 \\ 0 & -0.5 & 0.3 \\ 0 & 0.1 & -0.6 \end{bmatrix} \begin{array}{l} \\ \\ 0.5 R_3 + R_2 \end{array}$$

$$\begin{bmatrix} -0.3 & 0.1 & 0 \\ 0 & -0.5 & 0.3 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{r} -\frac{3}{10} x_1 + \frac{1}{10} x_2 = 0 \\ -\frac{1}{20} x_2 + \frac{3}{10} x_1 = 0 \\ 0 = 0 \end{array}$$

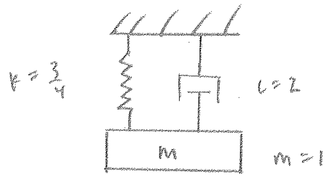
$\Rightarrow \begin{array}{l} x_1 = \frac{1}{3} x_2 \\ x_2 = 6 x_3 \end{array}$ AN EIGENVECTOR FOR $\lambda = 1$ IS $\begin{bmatrix} 2 \\ 6 \\ 1 \end{bmatrix}$

∴ IN THE LONG RUN, THE RATIO C : I : R WILL APPROACH 2 : 6 : 1 OR 22.22% COMMERCIAL, 66.67% INDUSTRIAL, 11.11% RESIDENTIAL

ANS

4. Find the general equation of motion for the following spring-mass system:

$$y''+2y'+\frac{3}{4}y=0$$



LET $y = y_1$ AND $y_1' = y_2$

$$\Rightarrow y_2' = y_1'' = -2y_1' - \frac{3}{4}y_1$$
$$= -2y_2 - \frac{3}{4}y_1$$

$$\Rightarrow \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{3}{4} & -2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\vec{y}\,' = A \quad \vec{y}$$

AS W/ OUR MIXING PROBLEM EXAMPLE, ASSUME $\vec{y} = \vec{x}e^{\lambda t} \Rightarrow \vec{y}\,' = \lambda\vec{x}e^{\lambda t}$

$$\Rightarrow \vec{y}\,' = \lambda\vec{x}e^{\lambda t} = A\vec{y} = A\vec{x}e^{\lambda t}$$

$$\lambda\vec{x}\cancel{e^{\lambda t}} = A\vec{x}\cancel{e^{\lambda t}} \Rightarrow A\vec{x} = \lambda\vec{x}$$ AN EIGENVALUE PROBLEM!

$$|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} 0-\lambda & 1 \\ -\frac{3}{4} & -2-\lambda \end{vmatrix} = -\lambda(-2-\lambda) + 0.75 = 0$$

$$\lambda^2 + 2\lambda + 0.75 = 0$$

$$(\lambda+0.5)(\lambda+1.5) = 0 \Rightarrow \lambda = -0.5,\ -1.5$$

IT CAN BE SHOWN THAT $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ IS AN EIGENVECTOR CORRESPONDING TO $\lambda = -0.5$ AND $\begin{bmatrix} 1 \\ -1.5 \end{bmatrix}$ IS AN EIGENVECTOR CORRESPONDING TO $\lambda = -1.5$

$\Rightarrow \vec{y} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}e^{-0.5t}$ AND $\vec{y} = \begin{bmatrix} 1 \\ -1.5 \end{bmatrix}e^{-1.5t}$ THE GENERAL SOLUTION IS A LINEAR COMBINATION OF THESE TWO

$$\therefore \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1\begin{bmatrix} -2 \\ 1 \end{bmatrix}e^{-0.5t} + c_2\begin{bmatrix} 1 \\ -1.5 \end{bmatrix}e^{-1.5t}$$

OR

$$y_1 = -2c_1e^{-0.5t} + c_2e^{-1.5t}$$
$$y_2 = c_1e^{-0.5t} - 1.5c_2e^{-1.5t}$$

ANS