LESSON 10 EXAMPLE PROBLEMS (SOLVING ODEs w/ LAPLACE TRANSFORMS)

① Solve $y'' + 2y' + 2y = 0$, $y(0) = 1$, $y'(0) = -3$ using laplace transforms

$$\mathcal{L}\{y'' + 2y' + 2y = 0\} = \mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{0\}$$

$$s^2 Y(s) - s\,y(0) - y'(0) + 2\{s\,Y(s) - y(0)\} + 2\,Y(s) = 0$$

$$Y(s)(s^2 + 2s + 2) - s(1) - (-3) - 2(1) = 0$$

$$Y(s) = \frac{s-1}{s^2+2s+2} = \frac{s-1}{(s+1)^2+1} = \frac{(s+1) - 2}{(s+1)^2+1} = \frac{s+1}{(s+1)^2+1} - \frac{2}{(s+1)^2+1}$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{s-1}{s^2+2s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$$

$$= e^{-t}\cos t - 2e^{-t}\sin t$$

ANS

② Solve $y'' - \frac{1}{4}y = 0$, $y(0) = 4$, $y'(0) = 0$ using laplace transforms

$$\mathcal{L}\{y'' - \tfrac{1}{4}y = 0\} = \mathcal{L}\{y''\} - \tfrac{1}{4}\mathcal{L}\{y\} = \mathcal{L}\{0\}$$

$$s^2 Y(s) - s\,y(0) - y'(0) - \tfrac{1}{4}Y(s) = 0$$

$$Y(s)(s^2 - \tfrac{1}{4}) - 4s = 0$$

$$Y(s) = \frac{4s}{s^2 - \frac{1}{4}} = 4\left(\frac{s}{s^2 - \frac{1}{4}}\right)$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{4s}{s^2 - \frac{1}{4}}\right\} = 4\mathcal{L}^{-1}\left\{\frac{s}{s^2 - \frac{1}{4}}\right\} = 4\cos\tfrac{1}{2}t$$

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③ Solve $y'' + 7y' + 12y = 21e^{3t}$, $y(0) = 3.5$, $y'(0) = -10$ using Laplace transforms

$$\mathcal{L}\{y'' + 7y' + 12y = 21e^{3t}\} = \mathcal{L}\{y''\} + 7\mathcal{L}\{y'\} + 12\mathcal{L}\{y\} = 21\mathcal{L}\{e^{3t}\}$$

$$s^2Y(s) - sy(0) - y'(0) + 7(sY(s) - y(0)) + 12Y(s) = 21\left(\frac{1}{s-3}\right)$$

$$Y(s)(s^2 + 7s + 12) - 3.5s + 10 - 24.5 = \frac{21}{s-3}$$

$$Y(s)(s^2 + 7s + 12) = \frac{21}{s-3} + 3.5s + 14.5$$

$$\Rightarrow Y(s) = \frac{21}{(s-3)(s^2+7s+12)} + 3.5\frac{s}{(s^2+7s+12)} + 14.5\left(\frac{1}{s^2+7s+12}\right)$$

$$= \underset{③}{\frac{21}{(s-3)(s+4)(s+3)}} + \underset{①}{\frac{3.5s}{(s+4)(s+3)}} + \underset{②}{\frac{14.5}{(s+4)(s+3)}}$$

$\boxed{①}$ $$\frac{3.5s}{(s+4)(s+3)} = \frac{A}{s+4} + \frac{B}{s+3}$$

MULTIPLY THROUGH BY $(s+4)$: $\frac{3.5s}{s+3} = A + \frac{B(s+4)}{s+3}$ LET $s = -4$

$$A = \frac{-4(3.5)}{-4+3} = 14$$

MULTIPLY THROUGH BY $(s+3)$: $\frac{3.5s}{s+4} = \frac{A(s+3)}{s+4} + B$ LET $s = -3$

$$\frac{-3(3.5)}{-3+4} = -10.5$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{3.5s}{(s+4)(s+3)}\right\} = \mathcal{L}^{-1}\left\{\frac{14}{s+4} - \frac{10.5}{s+3}\right\} = 14\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} - 10.5\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$$

$$= 14e^{-4t} - 10.5e^{-3t}$$

$\boxed{②}$ $$\frac{14.5}{(s+4)(s+3)} = \frac{A}{s+4} + \frac{B}{s+3}$$

MULTIPLY THROUGH BY $(s+4)$: $\frac{14.5}{s+3} = A + \frac{B(s+4)}{s+3}$ LET $s = -4$

$$A = \frac{14.5}{-4+3} = -14.5$$

MULTIPLY THROUGH BY $(s+3)$: $\frac{14.5}{s+4} = \frac{A(s+3)}{s+4} + B$ LET $s = -3$

$$B = \frac{14.5}{-3+4} = 14.5$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{14.5}{(s+4)(s+3)}\right\} = \mathcal{L}^{-1}\left\{\frac{-14.5}{s+4} + \frac{14.5}{s+3}\right\} = -14.5\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} + 14.5\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$$

$$= -14.5e^{-4t} + 14.5e^{-3t}$$

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SUMMING ① AND ② TOGETHER YIELDS $-\frac{1}{2}e^{-4t} + 4e^{-3t}$

$\boxed{③}$ $\frac{21}{(s-3)(s+4)(s+3)} = \frac{A}{s+4} + \frac{B}{s+3} + \frac{C}{s-3}$

MULTIPLY THROUGH BY $(s+4)$: $\frac{21}{(s-3)(s+3)} = A + \frac{B(s+4)}{s+3} + \frac{C(s+4)}{s-3}$ LET $s=-4$

$$A = \frac{21}{(-7)(-1)} = 3$$

MULTIPLY THROUGH BY $(s+3)$: $\frac{21}{(s-3)(s+4)} = \frac{A(s+3)}{s+4} + B + \frac{C(s+3)}{s-3}$ LET $s=-3$

$$B = \frac{21}{(-6)(1)} = \frac{-21}{6} = -\frac{7}{2}$$

MULTIPLY THROUGH BY $(s-3)$: $\frac{21}{(s+3)(s+4)} = \frac{A(s-3)}{s+4} + \frac{B(s-3)}{s+3} + C$ LET $s=3$

$$C = \frac{21}{(6)(7)} = \frac{21}{42} = \frac{1}{2}$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{21}{(s-3)(s+4)(s+3)}\right\} = \mathcal{L}^{-1}\left\{\frac{3}{s+4} - \frac{7}{2}\left(\frac{1}{s+3}\right) + \frac{1}{2}\left(\frac{1}{s-3}\right)\right\}$$

$$= 3\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} - \frac{7}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}$$

$$= 3e^{-4t} - \frac{7}{2}e^{-3t} + \frac{1}{2}e^{3t}$$

$$\Rightarrow \mathcal{L}^{-1}\{Y(s)\} = -\frac{1}{2}e^{-4t} + 4e^{-3t} + 3e^{-4t} - \frac{7}{2}e^{-3t} + \frac{1}{2}e^{3t}$$

$$= 2.5e^{-4t} + 0.5e^{-3t} + 0.5e^{3t}$$

ANS

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④ Solve $y'' + 5y' + 4y = 2\cos 4t$, $y(0) = 1$, $y'(0) = -1$ using Laplace transforms

$$\mathcal{L}\{y'' + 5y' + 4y = 2\cos 4t\} = \mathcal{L}\{y''\} + 5\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = 2\mathcal{L}\{\cos 4t\}$$

$$s^2 Y(s) - sy(0) - y'(0) + 5(sY(s) - y(0)) + 4Y(s) = 2\left(\frac{s}{s^2+16}\right)$$

$$Y(s)(s^2 + 5s + 4) - s(1) - (-1) - 5(1) = \frac{2s}{s^2+16}$$

$$Y(s)(s^2 + 5s + 4) = \frac{2s}{s^2+16} + s + 4$$

$$Y(s) = \frac{2s}{(s^2+16)(s^2+5s+4)} + \frac{s+4}{(s^2+5s+4)} = \frac{2s}{(s^2+16)(s+1)(s+4)} + \frac{\cancel{s+4}}{(s+1)\cancel{(s+4)}}$$

$$= \underbrace{\frac{2s}{(s^2+16)(s+1)(s+4)}}_{②} + \underbrace{\frac{1}{s+1}}_{①}$$

$\boxed{①}$ $\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}$

$\boxed{②}$ $$\frac{2s}{(s^2+16)(s+1)(s+4)} = \frac{A}{s+1} + \frac{B}{s+4} + \frac{Cs+D}{s^2+16}$$

NOTE USE OF LINEAR FORM $Cs + D$ WHEN WE HAVE A QUADRATIC IN THE DENOMINATOR

FOR A, MULTIPLY THROUGH BY $s+1$ AND LET $s = -1 \Rightarrow A = \frac{-2}{51}$

SIMILARLY FOR B, MULTIPLY THROUGH BY $s+4$ AND LET $s = -4 \Rightarrow B = \frac{1}{12}$

FOR $Cs + D$, MULTIPLY THROUGH BY s^2+16 AND LET $s = 4i \Rightarrow$

$$4Ci + D = \frac{8i}{(4i+1)(4i+4)} = \frac{8i}{-12+20i}\left(\frac{-12-20i}{-12-20i}\right)$$

$$= \frac{160 - 96i}{544}$$

SET $4Ci$ EQUAL TO THE IMAGINARY PART AND D EQUAL TO THE REAL PART

$\Rightarrow 4Ci = \frac{-96i}{544}$ OR $C = -\frac{3}{68}$

$D = \frac{160}{544} = \frac{10}{34}$

ASK YOURSELF: WHY DID WE FACTOR THIS OUT?

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{2s}{(s^2+16)(s+1)(s+4)}\right\} = \mathcal{L}\left\{-\frac{2}{51}\left(\frac{1}{s+1}\right) + \frac{1}{12}\left(\frac{1}{s+4}\right) - \frac{3}{68}\left(\frac{s}{s^2+16}\right) + \frac{10}{34}\left(\frac{1}{4}\right)\left(\frac{4}{s^2+16}\right)\right\}$$

$$= -\frac{2}{51}e^{-t} + \frac{1}{12}e^{-4t} - \frac{3}{68}\cos 4t + \frac{10}{136}\sin 4t$$

$$\Rightarrow \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{①\} + \mathcal{L}^{-1}\{②\} = \frac{49}{51}e^{-t} + \frac{1}{12}e^{-4t} - \frac{3}{68}\cos 4t + \frac{10}{136}\sin 4t$$

ANS

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