LESSON 12 EXAMPLE PROBLEMS (SHORT IMPULSES, DIRAC DELTA FUNCTION)

① Solve $y'' + 3y' + 2y = \delta(t-1)$, $y(0)=0$, $y'(0)=0$

$$\mathcal{L}\{y'' + 3y' + 2y\} = \mathcal{L}\{\delta(t-1)\}$$

$$s^2 Y(s) - s\cancel{y(0)}^{0} - \cancel{y'(0)}^{0} + 3sY(s) - 3\cancel{y(0)}^{0} + 2Y(s) = e^{-s}$$

$$Y(s)(s^2+3s+2) = e^{-s} \quad \Rightarrow \quad Y(s) = \frac{e^{-s}}{(s^2+3s+2)} = \frac{e^{-s}}{(s+1)(s+2)}$$

WE WANT THE RESPONSE $y(t) = \mathcal{L}^{-1}\{Y(s)\}$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{e^{-s}}{(s+1)(s+2)}\right\} = \mathcal{L}^{-1}\left\{e^{-s}\left(\frac{1}{(s+1)(s+2)}\right)\right\}$$

CONSIDER $\dfrac{1}{(s+1)(s+2)} = \dfrac{A}{s+1} + \dfrac{B}{s+2}$

MULTIPLY THRU BY $(s+1)$: $\dfrac{1}{s+2} = A + \dfrac{B(s+1)}{s+2}$

SETTING $s=-1$ YIELDS $A = \dfrac{1}{-1+2} = 1$

NOW MULTIPLY THRU BY $(s+2)$: $\dfrac{1}{s+1} = \dfrac{(1)(s+2)}{s+1} + B$

SETTING $s=-2$ YIELDS $B = \dfrac{1}{-2+1} = -1$

$$\Rightarrow \mathcal{L}^{-1}\left\{e^{-s}\left(\frac{1}{(s+1)(s+2)}\right)\right\} = \mathcal{L}^{-1}\left\{e^{-s}\left(\frac{1}{s+1} - \frac{1}{s+2}\right)\right\}$$

BY THE t-SHIFTING THEOREM, $\mathcal{L}^{-1}\{e^{-as}F(s)\} = u(t-a)\,f(t-a)$ WHERE $f(t) = \mathcal{L}^{-1}\{F(s)\}$

IN OUR EXAMPLE, $a=1$ AND $F(s) = \dfrac{1}{s+1} - \dfrac{1}{s+2}$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+1} - \frac{1}{s+2}\right\} = e^{-t} - e^{-2t} \quad \Rightarrow \quad f(t-1) = e^{-(t-1)} - e^{-2(t-1)}$$

$$\therefore \quad y(t) = \mathcal{L}^{-1}\{Y(s)\} = u(t-1)\left[e^{-(t-1)} - e^{-2(t-1)}\right]$$

ANS

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① Solve $y'' + 5y = 25t - 100\,\delta(t-\pi)$, $y(0) = -2$, $y'(0) = 5$

$$\mathcal{L}\{y'' + 5y\} = \mathcal{L}\{25t - 100\,\delta(t-\pi)\}$$

$$s^2 Y(s) - s\,y(0) - y'(0) + 5Y(s) = \frac{25}{s^2} - 100\,e^{-\pi s}$$

$$Y(s)(s^2+5) + 2s - 5 = \frac{25}{s^2} - 100\,e^{-\pi s}$$

$$\Rightarrow Y(s) = \frac{-2s}{s^2+5} + \frac{5}{s^2+5} + \frac{25}{s^2(s^2+5)} - \frac{100\,e^{-\pi s}}{s^2+5}$$

WE WANT THE RESPONSE $y(t) = \mathcal{L}^{-1}\{Y(s)\}$

$$\mathcal{L}^{-1}\{Y(s)\} = \underbrace{-2\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+5}\right\}}_{①} + \underbrace{\sqrt{5}\,\mathcal{L}^{-1}\left\{\frac{\sqrt{5}}{s^2+5}\right\}}_{②} + \underbrace{\mathcal{L}^{-1}\left\{\frac{25}{s^2(s^2+5)}\right\}}_{③} \underbrace{-100\,\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+5}\right\}}_{④}$$

$\boxed{①}$ $-2\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+5}\right\} = -2\cos\sqrt{5}\,t$

$\boxed{②}$ $\sqrt{5}\,\mathcal{L}^{-1}\left\{\frac{\sqrt{5}}{s^2+5}\right\} = \sqrt{5}\sin\sqrt{5}\,t$

$\boxed{④}$ $-100\,\mathcal{L}^{-1}\left\{e^{-\pi s}\left(\frac{1}{s^2+5}\right)\right\} = -100\left(\frac{1}{\sqrt{5}}\right)\mathcal{L}^{-1}\left\{e^{-\pi s}\left(\frac{\sqrt{5}}{s^2+5}\right)\right\} = \frac{-100}{\sqrt{5}}\,u(t-\pi)\sin\sqrt{5}(t-\pi)$

NOTE USE OF t-SHIFTING THM. HERE!

$\boxed{③}$ $\mathcal{L}^{-1}\left\{\frac{25}{s^2(s^2+5)}\right\}$

CONSIDER $\frac{25}{s^2(s^2+5)} = \frac{As+B}{s^2} + \frac{Cs+D}{s^2+5}$

MULTIPLY THROUGH BY $s^2(s^2+25)$: $25 = (As+B)(s^2+5) + (Cs+D)(s^2)$

$$25 = As^3 + Bs^2 + 5As + 5B + Cs^3 + Ds^2$$

EQUATING LIKE POWERS OF s ON EITHER SIDE YIELDS

$$25 = 5B \Rightarrow B = 5$$
$$0s = (5A)s \Rightarrow A = 0$$
$$0s^2 = (B+D)s = (5+D)s \Rightarrow D = -5$$
$$0s^3 = (A+C)s^3 = (0+C)s^3 \Rightarrow C = 0$$

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$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{25}{s^2(s^2+5)}\right\} = \mathcal{L}^{-1}\left\{\frac{5}{s^2} - \frac{5}{s^2+25}\right\}$$

$$= 5\,\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \sqrt{5}\,\mathcal{L}^{-1}\left\{\frac{\sqrt{5}}{s^2+25}\right\}$$

$$= 5t - \sqrt{5}\sin\sqrt{5}t$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\{Y(s)\} = -2\cos\sqrt{5}t + \cancel{\sqrt{5}\sin\sqrt{5}t} - \frac{100}{\sqrt{5}}\,u(t-\pi)\sin\sqrt{5}(t-\pi) + 5t - \cancel{\sqrt{5}\sin\sqrt{5}t}$$

$$= -2\cos\sqrt{5}t + 5t - \frac{100}{\sqrt{5}}\,u(t-\pi)\sin\sqrt{5}(t-\pi)$$

Ans

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