LESSON 16 (OVER/UNDER DETERMINED SYSTEMS, DETERMINANTS, INVERSES) EXAMPLE PROBLEMS

① Example of a linear system w/ infinitely many solutions

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 4 \\ 6x_1 + x_2 + 2x_3 &= 6 \end{aligned} \xrightarrow{\text{AUGMENTED MATRIX}} \left[\begin{array}{ccc|c} 3 & 2 & 1 & 4 \\ 6 & 1 & 2 & 6 \end{array}\right] \quad R_2 - 2R_1$$

$$= \left[\begin{array}{ccc|c} 3 & 2 & 1 & 4 \\ 0 & -3 & 0 & -2 \end{array}\right] \quad \text{MULTIPLY } R_2 \text{ BY } (-1)$$

$$= \left[\begin{array}{ccc|c} 3 & 2 & 1 & 4 \\ 0 & 3 & 0 & 2 \end{array}\right]$$

$$\Rightarrow \begin{aligned} 3x_1 + 2x_2 + x_3 &= 4 \\ 3x_2 &= 2 \end{aligned}$$

BACK SUBSTITUTION YIELDS $x_2 = \frac{2}{3}$

$x_1 = \frac{8}{9} - \frac{1}{3}x_3$

WE HAVE A "DEGREE OF FREEDOM" w/ OUR CHOICE OF x_3 $\longrightarrow$ INFINITE CHOICES FOR x_3 YIELDS AN INFINITE # OF POSSIBLE SOLUTIONS

$\Rightarrow$ "A" SOLUTION IS $x_1 = \frac{8}{9}$, $x_2 = \frac{2}{3}$, $x_3 = 0$

② Example of a linear system w/ no solutions

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 3 \\ 2x_1 + x_2 + x_3 &= 0 \\ 6x_1 + 2x_2 + 4x_3 &= 6 \end{aligned} \xrightarrow{\text{AUGMENTED MATRIX}} \left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{array}\right] \quad \begin{array}{l} R_2 - \frac{2}{3}R_1 \\ R_3 - 2R_1 \end{array}$$

$$= \begin{bmatrix} 3 & 2 & 1 & 3 \\ 0 & -\frac{1}{3} & \frac{1}{3} & -2 \\ 0 & -2 & 2 & 0 \end{bmatrix} \quad R_3 - 6R_2$$

$$= \begin{bmatrix} 3 & 2 & 1 & 3 \\ 0 & -\frac{1}{3} & \frac{1}{3} & -2 \\ 0 & 0 & 0 & 12 \end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned} 3x_1 + 2x_2 + x_3 &= 3 \\ -\tfrac{1}{3}x_2 + \tfrac{1}{3}x_3 &= -2 \\ 0 &= 12 \end{aligned}$$

$\longleftarrow$ THIS IS A CONTRADICTION!

$\Rightarrow$ NO SOLUTIONS TO THIS SYSTEM

③ Find the determinant of $A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{bmatrix}$

$$D = \det A = \begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix}$$

WE WILL DO A COFACTOR EXPANSION ON THE FIRST ROW:

$$\begin{vmatrix} 1 & \cancel{3} & \cancel{0} \\ \cancel{2} & 6 & 4 \\ \cancel{-1} & 0 & 2 \end{vmatrix} \qquad (1)\cdot\begin{vmatrix} 6 & 4 \\ 0 & 2 \end{vmatrix} = (1)(6\cdot 2 - 4\cdot 0) = 12$$

SIGN? $(-1)^{1+1} = (-1)^2 = +1$

NOTE: OUR SIGN CONVENTION COMES FROM $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$ OR $(-1)^{j+k}$ WHERE OUR CHOSEN ELEMENT IS IN THE jth ROW, kth COLUMN

$$\begin{vmatrix} \cancel{1} & 3 & \cancel{0} \\ 2 & \cancel{6} & 4 \\ -1 & \cancel{0} & 2 \end{vmatrix} \qquad (3)\cdot\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix} = 3\,(2\cdot 2 - 4(-1)) = 24$$

SIGN $= (-1)^{1+2} = (-1)^3 = -1$

$$\begin{vmatrix} \cancel{1} & \cancel{3} & 0 \\ 2 & 6 & \cancel{4} \\ -1 & 0 & \cancel{2} \end{vmatrix} \qquad (0)\cdot\begin{vmatrix} 2 & 6 \\ -1 & 0 \end{vmatrix} = 0\,(2\cdot 0 - 6(-1)) = 0$$

SIGN $= (-1)^{1+3} = (-1)^4 = +1$

ADDING THESE TOGETHER YIELDS $D = +12 - 24 + 0 = -12$

(4) Find the determinant of $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix}$

This example serves to illustrate how we can use the structure of the matrix to our advantage when calculating determinants

Recall that we can do a cofactor expansion on ANY row or column

Let's pick the second row :

$$D = \det A = \begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 3 & 1 & 1 \end{vmatrix} \Rightarrow \begin{vmatrix} 2 & 1 & 1 \\ \boxed{0 \quad 1 \quad 0} \\ 3 & 1 & 1 \end{vmatrix}$$

Since two of the elements in this row are zero, we only have to calculate the cofactor associated w/ 1

$$\Rightarrow \begin{vmatrix} 2 & \cancel{1} & 1 \\ \cancel{0} & 1 & \cancel{0} \\ 3 & \cancel{1} & 1 \end{vmatrix} \qquad (1) \cdot \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} = (1)(2 \cdot 1 - 3 \cdot 1) = -1$$

$$\text{SIGN} = (-1)^{2+2} = (-1)^4 = +1$$

$$\Rightarrow D = (+) - 1 = \underline{\underline{-1}}$$

ANS

⑤ Find the inverse A^{-1} of $A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$

USING THE FORMULA FOR A 2x2 MATRIX: $A^{-1} = \frac{1}{\det A}\begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$

$$= \frac{1}{2\cdot3 - 1\cdot1}\begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix}$$

USING GAUSS-JORDAN ELIMINATION:

$$\left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right] R_2 - \tfrac{1}{2}R_1 = \left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 0 & \frac{5}{2} & -\frac{1}{2} & 1 \end{array}\right] \tfrac{2}{5}R_2 = \left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 0 & 1 & -\frac{1}{5} & \frac{2}{5} \end{array}\right] R_1 - R_2$$

$$= \left[\begin{array}{cc|cc} 2 & 0 & \frac{6}{5} & -\frac{2}{5} \\ 0 & 1 & -\frac{1}{5} & \frac{2}{5} \end{array}\right] \tfrac{1}{2}R_1 = \left[\begin{array}{cc|cc} 1 & 0 & \frac{3}{5} & -\frac{1}{5} \\ 0 & 1 & -\frac{1}{5} & \frac{2}{5} \end{array}\right]$$

$$\Rightarrow A^{-1} = \begin{bmatrix} \frac{3}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix}$$

Recall our methodology w/ Gauss-Jordan Elimination:

① CREATE THE AUGMENTED MATRIX $[A \mid I]$

② PERFORM ELEMENTARY ROW OPERATIONS UNTIL A IS THE IDENTITY MATRIX: $[I \mid A^*]$

③ A^{-1} IS NOW WHERE THE IDENTITY MATRIX WAS $\Rightarrow A^* = A^{-1}$