LESSON 17 (EIGENVALUES AND EIGENVECTORS) EXAMPLE PROBLEMS

① Find the eigenvalues and eigenvectors of $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix}$$

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)(1-\lambda) - 2 \cdot 2 = 0$$

$$1 - 2\lambda + \lambda^2 - 4 = 0$$
$$\lambda^2 - 2\lambda - 3 = 0$$
$$(\lambda+1)(\lambda-3) = 0$$

$\Rightarrow \lambda = -1, 3$

what are the corresponding eigenvectors?

$\boxed{\lambda = -1}$ Solve $(A - \lambda I)\vec{x} = \vec{0}$

$$\Rightarrow \begin{bmatrix} 1-(-1) & 2 \\ 2 & 1-(-1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

GAUSSIAN ELIMINATION $\rightarrow$ $R_2 - R_1$

$$\Rightarrow \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2x_1 + 2x_2 = 0$$
$$0 = 0 \quad \Rightarrow \quad x_1 = -x_2$$

Note that eigenvectors are not unique, and this analysis only determines an eigenvector of $\lambda = -1$ up to a scalar multiple ... we only need one such eigenvector

$\Rightarrow$ AN EIGENVECTOR is $x_1 = 1$, $x_2 = -1$ $\Rightarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ FOR $\lambda = -1$

$\boxed{\lambda = -3}$ Solve $(A - \lambda I)\vec{x} = \vec{0}$

$$\Rightarrow \begin{bmatrix} 1-3 & 2 \\ 2 & 1-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

GAUSSIAN ELIMINATION $\rightarrow$ $R_2 + R_1$

$$\Rightarrow \begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-2x_1 + 2x_2 = 0$

$0 = 0 \quad \Rightarrow x_1 = x_2$

$\Rightarrow$ AN EIGENVECTOR is $x_1 = 1,\ x_2 = 1 \Rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ FOR $\lambda = -3$

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② Find the eigenvalues and eigenvectors of $\begin{bmatrix} 4 & -2 & 3 \\ -2 & 1 & 6 \\ 1 & 2 & 2 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 4 & -2 & 3 \\ -2 & 1 & 6 \\ 1 & 2 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4-\lambda & -2 & 3 \\ -2 & 1-\lambda & 6 \\ 1 & 2 & 2-\lambda \end{bmatrix}$$

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 4-\lambda & -2 & 3 \\ -2 & 1-\lambda & 6 \\ 1 & 2 & 2-\lambda \end{vmatrix} = +(4-\lambda)\begin{vmatrix} 1-\lambda & 6 \\ 2 & 2-\lambda \end{vmatrix} - (-2)\begin{vmatrix} -2 & 6 \\ 1 & 2-\lambda \end{vmatrix} + 3\begin{vmatrix} -2 & 1-\lambda \\ 1 & 2 \end{vmatrix}$$

$$= (4-\lambda)\left((1-\lambda)(2-\lambda) - 6\cdot 2\right) + 2\left(-2(2-\lambda) - 6\cdot 1\right) + 3\left(-2\cdot 2 - 1(1-\lambda)\right)$$

$$= (4-\lambda)\left(2 - 3\lambda + \lambda^2 - 12\right) + 2\left(-4 + 2\lambda - 6\right) + 3\left(-4 - 1 + \lambda\right)$$

$$= (4-\lambda)\left(\lambda^2 - 3\lambda - 10\right) + 2\left(2\lambda - 10\right) + 3\left(\lambda - 5\right)$$

$$= 4\lambda^2 - 12\lambda - 40 - \lambda^3 + 3\lambda^2 + 10\lambda + 4\lambda - 20 + 3\lambda - 15$$

$$= -\lambda^3 + 7\lambda^2 + 5\lambda - 75 = 0$$

LOOK @ THE INTEGER FACTORS OF -75 AND TEST TO FIND A ROOT OF THIS EXPRESSION $(\pm 3, \pm 5, \pm 15, \pm 25)$

$+3$: $\quad -(3)^3 + 7(3)^2 + 5(3) - 75 \stackrel{?}{=} 0$

$$-27 + 63 + 15 - 75 = -24 \neq 0 \qquad \text{NO!}$$

-3: $\quad -(-3)^3 + 7(-3)^2 + 5(-3) - 75 \stackrel{?}{=} 0$

$$27 + 63 - 15 - 75 = 0 \qquad \text{YES!}$$

$\Rightarrow$ ONE EIGENVALUE IS $\lambda = -3$, USE SYNTHETIC DIVISION TO FIND THE OTHER TWO

$$\begin{array}{r|l}
 & -\lambda^2 + 10\lambda - 25 \\
\hline
\lambda + 3 & -\lambda^3 + 7\lambda^2 + 5\lambda - 75 \\
 & (-)\ -\lambda^3 - 3\lambda^2 \\
\hline
 & \quad 10\lambda^2 + 5\lambda \\
 & (-)\ 10\lambda^2 + 30\lambda \\
\hline
 & \qquad -25\lambda - 75 \\
 & (-)\ -25\lambda - 75 \\
\hline
 & \qquad\quad 0
\end{array}$$

$\Rightarrow -\lambda^2 + 7\lambda^2 + 5\lambda - 75 = (\lambda+3)(-\lambda^2 + 10\lambda - 25)$

$$= (\lambda+3)(-\lambda+5)(\lambda-5) = 0$$

$\Rightarrow$ THE OTHER TWO EIGENVALUES ARE $\lambda = 5, 5$ NOTE MULTIPLICITY!

What are the corresponding eigenvectors?

$\boxed{\lambda = -3}$ Solve $(A - \lambda I)\vec{x} = \vec{0}$

$$\Rightarrow \begin{bmatrix} 4-(-3) & -2 & 3 \\ -2 & 1-(-3) & 6 \\ 1 & 2 & 2-(-3) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 7 & -2 & 3 \\ -2 & 4 & 6 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{array}{l} \frac{7}{2}R_2 + R_1 \\ 7R_3 - R_1 \end{array}$$

$$\begin{bmatrix} 7 & -2 & 3 \\ 0 & 12 & 24 \\ 0 & 16 & 32 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad -\frac{12}{16}R_3 + R_2$$

$$\begin{bmatrix} 7 & -2 & 3 \\ 0 & 12 & 24 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned} 7x_1 - 2x_2 + 3x_3 &= 0 \\ 12x_2 + 24x_3 &= 0 \\ 0 &= 0 \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} x_1 &= \frac{-4x_3 + 3x_3}{7} = -x_3 \\ x_2 &= -2x_3 \end{aligned}$$

$\Rightarrow$ AN EIGENVECTOR IS $x_1 = -1,\ x_2 = -2,\ x_3 = 1 \Rightarrow \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$ FOR $\lambda = -3$

$\boxed{\lambda = 5}$ Solve $(A - \lambda I)\vec{x} = \vec{0}$

$$\Rightarrow \begin{bmatrix} 4-5 & -2 & 3 \\ -2 & 1-5 & 6 \\ 1 & 2 & 2-5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -2 & 3 \\ -2 & -4 & 6 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{matrix} R_2 - 2R_1 \\ R_3 + R_1 \end{matrix}$$

$$\begin{bmatrix} -1 & -2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned} -x_1 - 2x_2 + 3x_3 &= 0 \\ 0 &= 0 \\ 0 &= 0 \end{aligned} \qquad \Rightarrow \qquad x_1 = -2x_2 + 3x_3$$

Since we have two free variables x_2, x_3 we can get two LINEARLY INDEPENDENT eigenvectors for $\lambda = 5$

$\Rightarrow \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ are two linearly independent eigenvectors