EXAMPLE PROBLEMS, LESSON #25 (PATH INDEPENDENT LINE INTEGRALS)

1. $\int_{(5,0)}^{(0,5)} \left(\underset{F_1}{y^2 e^{2x}}\, dx + \underset{F_2}{y e^{2x}}\, dy \right)$

CHECK FOR EXACTNESS: $\frac{\partial F_2}{\partial x} \overset{?}{=} \frac{\partial F_1}{\partial y} \Rightarrow 2ye^{2x} = 2ye^{2x}$ ✓

PATH INDEPENDENT $\rightarrow$ $\vec{F} = \langle y^2 e^{2x}, ye^{2x} \rangle = \text{grad } f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$

$\frac{\partial f}{\partial x} = F_1 = y^2 e^{2x} \Rightarrow f = \int y^2 e^{2x}\, dx$

$= \frac{1}{2} y^2 e^{2x} + g(y)$

$\frac{\partial f}{\partial y} = ye^{2x} + g'(y) = F_2 = ye^{2x} \Rightarrow g'(y) = 0$ OR $g(y) = C$

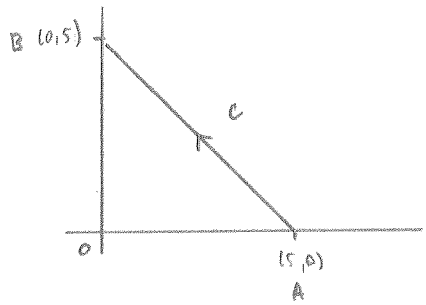
$\therefore f(x,y) = \frac{1}{2} y^2 e^{2x} + C$

$\Rightarrow \int_{(5,0)}^{(0,5)} (y^2 e^{2x} dx + ye^{2x} dy) = f(0,5) - f(5,0)$

$= \frac{1}{2}(5)^2 e^{2(0)} - \frac{1}{2}(0)^2 e^{2(5)}$

$= \frac{25}{2} = \underline{\underline{12.5}}$

2. WE DO THE SAME PROBLEM AS ABOVE, EXCEPT WE WILL CALCULATE $\int_C \vec{F}(\vec{r}) \cdot \vec{r}\,'(t)\, dt$ OVER MULTIPLE PATHS TO DEMONSTRATE PATH INDEPENDENCE



$\vec{r}(t) = \langle t, -t+5 \rangle$; $\vec{r}(5) = \langle 5, 0 \rangle$, $\vec{r}(0) = \langle 0, 5 \rangle$

$\vec{F} = \langle y^2 e^{2x}, ye^{2x} \rangle \Rightarrow \vec{F}(\vec{r}) = \langle (-t+5)^2 e^{2t}, (-t+5) e^{2t} \rangle$

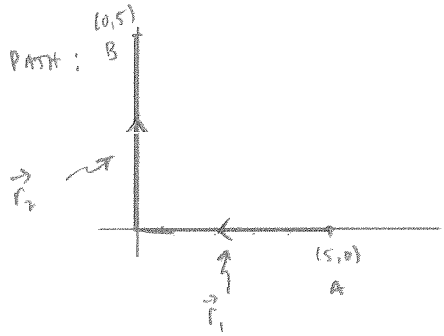
$\vec{r}\,'(t) = \langle 1, -1 \rangle$

$\int_C \vec{F}(\vec{r}) \cdot \vec{r}\,'(t)\, dt = \int_5^0 \langle (-t+5)^2 e^{2t}, (-t+5) e^{2t} \rangle \cdot \langle 1, -1 \rangle\, dt$

$= \int_5^0 \left((-t+5)^2 e^{2t} - (-t+5) e^{2t} \right) dt$

$= 12.5$ (Mathematica)

ANOTHER PATH :



$\vec{r}_1(t) = \langle t, 0 \rangle \Rightarrow \vec{r}_1{}'(t) = \langle 1, 0 \rangle$

$(5,0) = \vec{r}_1(5)$ AND $(0,0) = \vec{r}_1(0)$

$\vec{r}_2(t) = \langle 0, t \rangle \Rightarrow \vec{r}_2{}'(t) = \langle 0, 1 \rangle$

$(0,0) = \vec{r}_2(0)$ AND $(0,5) = \vec{r}_1(5)$

$$\int_C \vec{F}(\vec{r}) \cdot \vec{r}\,'(t)\,dt = \int_5^0 \langle (0)^2 e^{2t}, (0) e^{2t} \rangle \cdot \langle 1, 0 \rangle\,dt + \int_0^5 \langle t^2 e^{2(0)}, t e^{2(0)} \rangle \cdot \langle 0, 1 \rangle\,dt$$

$$= \int_5^0 0\,dt + \int_0^5 t\,dt$$

$$= \tfrac{1}{2} t^2 \Big|_0^5 = \underline{\underline{12.5}}$$

3. $\displaystyle\int_{(0,0,0)}^{(1,1,1)} e^{x^2+y^2-2z}\,(x\,dx + y\,dy - dz)$

$F_1 = x e^{x^2+y^2-2z}$, $F_2 = y e^{x^2+y^2-2z}$, $F_3 = -e^{x^2+y^2-2z}$

CHECK FOR EXACTNESS : $\dfrac{\partial F_3}{\partial y} \overset{?}{=} \dfrac{\partial F_2}{\partial z} \Rightarrow -2y\, e^{x^2+y^2-2z} = -2y\, e^{x^2+y^2-2z}$ ✓

$\dfrac{\partial F_1}{\partial z} \overset{?}{=} \dfrac{\partial F_3}{\partial x} \Rightarrow -2x\, e^{x^2+y^2-2z} = -2x\, e^{x^2+y^2-2z}$ ✓

$\dfrac{\partial F_2}{\partial x} \overset{?}{=} \dfrac{\partial F_1}{\partial y} \Rightarrow 2xy\, e^{x^2+y^2-2z} = 2xy\, e^{x^2+y^2-2z}$ ✓

PATH INDEPENDENT $\rightarrow$ $\vec{F} = \langle x e^{x^2+y^2-2z}, y e^{x^2+y^2-2z}, -e^{x^2+y^2-2z} \rangle = \text{grad } f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$

$\dfrac{\partial f}{\partial x} = F_1 = x e^{x^2+y^2-2z} \Rightarrow f = \displaystyle\int x e^{x^2+y^2-2z}\,dx$

$u = x^2+y^2-2z$, $du = 2x\,dx$

$\int \frac{1}{2} e^u\,du = \frac{1}{2} e^u$

$\Rightarrow f = \frac{1}{2} e^{x^2+y^2-2z} + g(y,z)$

$\dfrac{\partial f}{\partial y} = \frac{1}{2}(2y)\, e^{x^2+y^2-2z} + \dfrac{\partial g}{\partial y} = F_2 = y e^{x^2+y^2-2z}$

$\Rightarrow \dfrac{\partial g}{\partial y} = 0$ OR $g(y,z) = h(z)$

$\Rightarrow f = \frac{1}{2} e^{x^2+y^2-2z} + h(z)$

$$\frac{\partial f}{\partial z} = \frac{1}{2}(-2)\, e^{x^2+y^2-2z} + h'(z) = F_3 = -e^{x^2+y^2-2z}$$

$$\Rightarrow h'(z) = 0 \quad \text{or} \quad h(z) = C$$

$$\therefore f(x,y,z) = \frac{1}{2} e^{x^2+y^2-2z} + C$$

$$\Rightarrow \int_{(0,0,0)}^{(1,1,1)} e^{x^2+y^2-2z}(x\,dx + y\,dy - dz) = \frac{1}{2} e^{x^2+y^2-2z} + C \Big|_{(0,0,0)}^{(1,1,1)}$$

$$= \frac{1}{2} e^{(1)^2+(1)^2-2(1)} + C - \frac{1}{2} e^{(0)^2+(0)^2-2(0)} - C$$

$$= \frac{1}{2} e^0 - \frac{1}{2} e^0 = \underline{\underline{0}}$$

4. $\int_{(2,3,0)}^{(0,1,2)} (z e^{xz}\,dx + dy + x e^{xz}\,dz)$

$F_1 = z e^{xz}$, $F_2 = 1$, $F_3 = x e^{xz}$

CHECK FOR EXACTNESS: $\frac{\partial F_3}{\partial y} \overset{?}{=} \frac{\partial F_2}{\partial z} \Rightarrow 0 = 0$ ✓

$\frac{\partial F_1}{\partial z} \overset{?}{=} \frac{\partial F_3}{\partial x} \Rightarrow e^{xz} + zxe^{xz} = e^{xz} + zxe^{xz}$ ✓

$\frac{\partial F_2}{\partial x} \overset{?}{=} \frac{\partial F_1}{\partial y} \Rightarrow 0 = 0$ ✓

PATH INDEPENDENT $\rightarrow$ $\vec{F} = \langle z e^{xz}, 1, x e^{xz} \rangle = \text{grad } f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$

$$\frac{\partial f}{\partial x} = F_1 = z e^{xz} \Rightarrow f = \int z e^{xz}\,dx$$

$$f = e^{xz} + g(y,z)$$

$$\frac{\partial f}{\partial y} = 0 + \frac{\partial g}{\partial y} = F_2 = 1 \Rightarrow g = \int dy$$

$$g = y + h(z)$$

$$\Rightarrow f = e^{xz} + y + h(z)$$

$$\frac{\partial f}{\partial z} = x e^{xz} + 0 + h'(z) = F_3 = x e^{xz}$$

$$\Rightarrow h'(z) = 0 \quad \text{or} \quad h(z) = C$$

$$\therefore f(x,y,z) = e^{xz} + y + C$$

$$\Rightarrow \int_{(2,3,0)}^{(0,1,2)} (z e^{xz}\,dx + dy + x e^{xz}\,dz) = f(0,1,2) - f(2,3,0)$$

$$= e^{xz} + y + C \Big|_{(2,3,0)}^{(0,1,2)} = \underline{\underline{-2}}$$