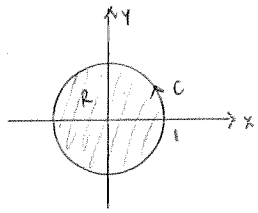
EXAMPLE PROBLEMS, LESSON #26 (GREEN'S THEOREM IN THE PLANE)

1. VERIFY GREEN'S THEOREM FOR $F_1 = y^2 - 7y$, $F_2 = 2xy + 2x$ AND C IS THE CIRCLE $x^2 + y^2 = 1$



$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = (2y+2) - (2y-7) = 9$$

GREEN'S THM: $\oint_C \vec{F} \cdot d\vec{r} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\, dy$

$$= \iint_R 9 \, dx\, dy$$

$$= 9 \iint_R dx\, dy = \underline{\underline{9\pi}}$$

AREA OF R IS $\pi r^2 = \pi (1)^2 = \pi$

CALCULATING $\oint_C \vec{F} \cdot d\vec{r}$ NORMALLY:

$$\oint_C \vec{F} \cdot d\vec{r} = \int_{t_0}^{t_f} \vec{F}(\vec{r}) \cdot \vec{r}\,'(t)\, dt$$

$\vec{r}(t) = \langle \cos t, \sin t \rangle \Rightarrow \vec{r}\,'(t) = \langle -\sin t, \cos t \rangle$

NOTE THAT C HAS START POINT AND ENDPOINT $(1,0)$

$\Rightarrow \vec{r}(0) = \vec{r}(2\pi) = \langle 0, 0 \rangle$ SO $t_0 = 0$, $t_f = 2\pi$

$\vec{F} = \langle F_1, F_2 \rangle = \langle y^2 - 7y, 2xy + 2x \rangle$

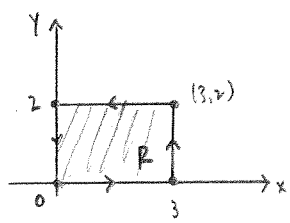
$\Rightarrow \vec{F}(\vec{r}) = \langle \sin^2 t - 7\sin t, 2\cos t \sin t + 2\cos t \rangle$ SINCE $x(t) = \cos t$, $y(t) = \sin t$

$$\Rightarrow \int_{t_0}^{t_f} \vec{F}(\vec{r}) \cdot \vec{r}\,'(t)\, dt = \int_0^{2\pi} \langle \sin^2 t - 7\sin t, 2\cos t \sin t + 2\cos t \rangle \cdot \langle -\sin t, \cos t \rangle\, dt$$

$$= \int_0^{2\pi} \left(-\sin^3 t + 7\sin^2 t + 2\cos^2 t \sin t + 2\cos^2 t \right) dt$$

$$= 0 + 7\pi - 0 + 2\pi = \underline{\underline{9\pi}}$$ (Mathematica)

2. EVALUATE $\oint_C \vec{F}(\vec{r}) \cdot d\vec{r}$ FOR $\vec{F} = \langle \frac{1}{2} x y^4, \frac{1}{2} x^4 y \rangle$, R IS THE RECTANGLE W/ VERTICES $(0,0)$, $(3,0)$, $(3,2)$ and $(0,2)$



NOTE THAT THE CALCULATION OF $\oint_C \vec{F}(\vec{r}) \cdot d\vec{r}$ WOULD TAKE FOUR INDIVIDUAL LINE INTEGRALS BUT CAN BE ACCOMPLISHED IN A SINGLE DOUBLE INTEGRAL USING GREEN'S THEOREM:

$$\oint_C \vec{F}(\vec{r}) \cdot d\vec{r} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\, dy$$

$\vec{F} = \langle F_1, F_2 \rangle = \langle \frac{1}{2} x y^4, \frac{1}{2} x^4 y \rangle \Rightarrow \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 2x^3 y - 2xy^3$

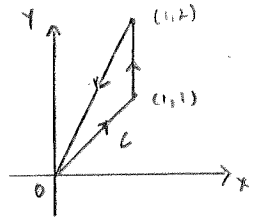
$$\Rightarrow \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy = \int_0^3\int_0^2 (2x^3y - 2xy^3)\,dy\,dx$$

$$= \int_0^3 \left[x^3y^2 - \tfrac{1}{2}xy^4\right]_{y=0}^{y=2} dx$$

$$= \int_0^3 (4x^3 - 8x)\,dx$$

$$= x^4 - 4x^2 \Big|_{x=0}^{x=3} = 81 - 4(9) = \underline{\underline{45}}$$

3. (§ 10.4, p.444 #5). USING GREEN'S THM, EVALUATE $\oint_C \vec{F}(\vec{r}) \cdot d\vec{r}$ COUNTER-CLOCKWISE AROUND THE BOUNDARY CURVE C OF THE REGION R WHERE $\vec{F} = \langle e^{x+y}, e^{x-y} \rangle$ AND R IS THE TRIANGLE w/ VERTICES (0,0), (1,1), (1,2)



$$\vec{F} = \langle F_1, F_2 \rangle = \langle e^{x+y}, e^{x-y} \rangle$$

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = e^{x-y} - e^{x+y}$$

$$\oint_C \vec{F}(\vec{r}) \cdot d\vec{r} = \int_0^1 \int_x^{2x} (e^{x-y} - e^{x+y})\,dy\,dx$$

$$= \int_0^1 \left[-e^{x-y} - e^{x+y}\right]_{y=x}^{y=2x} dx$$

$$= \int_0^1 (-e^{-x} - e^{3x} + e^0 + e^{2x})\,dx$$

$$= \left(e^{-x} - \tfrac{1}{3}e^{3x} + x + \tfrac{1}{2}e^{2x}\right)\Big|_{x=0}^{x=1}$$

$$= \frac{1}{e} - \frac{1}{3}e^3 + 1 + \frac{1}{2}e^2 - 1 + \frac{1}{3} - \frac{1}{2}$$

$$= \frac{1}{e} - \frac{1}{3}e^3 + \frac{1}{2}e^2 - \frac{1}{6}$$

ANS