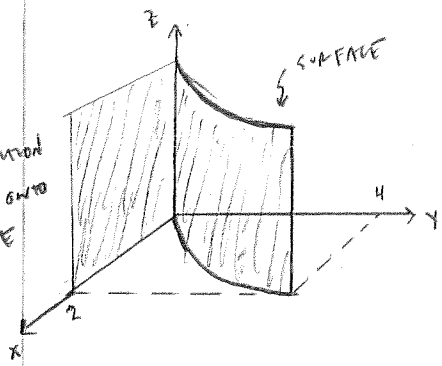
EXAMPLE PROBLEMS, LESSON 27 (SURFACE INTEGRALS)

1. FIND THE AMOUNT OF MATERIAL FLOWING THROUGH $S: y = x^2,\ 0 \le x \le 2,\ 0 \le z \le 3$ GIVEN $\vec{F} = \langle 3z^2, 6, 6xz \rangle$ (FLUX)



F: PROJECTION OF SURFACE ONTO XZ-PLANE

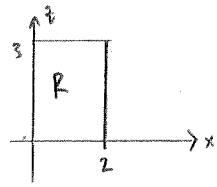
SINCE WE HAVE $y = f(x,z) = x^2$, WE PROJECT ONTO THE XZ-PLANE AND USE THE FORM

$$\iint_S \vec{F} \cdot \vec{n}\, dS = \iint_R \vec{F} \cdot \langle -f_x, 1, -f_z \rangle\, dx\, dz$$

$\vec{F} = \langle 3z^2, 6, 6xz \rangle$

$f(x,z) = y = x^2 \Rightarrow \langle -f_x, 1, -f_z \rangle = \langle -2x, 1, 0 \rangle$

LIMITS? OUR PROJECTION:



$0 \le z \le 3$
$0 \le x \le 2$

$$\Rightarrow \int_0^3 \int_0^2 \langle 3z^2, 6, 6xz \rangle \cdot \langle -2x, 1, 0 \rangle\, dx\, dz = \int_0^3 \int_0^2 (-6xz^2 + 6)\, dx\, dz$$

$$= \int_0^3 \left[-3x^2 z^2 + 6x \right]_{x=0}^{x=2} dz$$

$$= \int_0^3 (-12z^2 + 12)\, dz$$

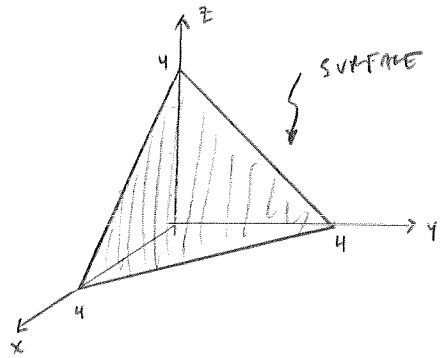
$$= -4z^3 + 12z \Big|_{x=0}^{x=3} = -72$$

NOTE THAT -72 IS ASSOCIATED W/ FLOW $(-)\,x$ DIRECTION, SO THIS CALCULATION USED THE NEGATIVE NORMAL TO OUR SURFACE

SINCE THE SURFACE IS ORIENTABLE, WE WANT THE POSITIVE NORMAL, SO OUR FINAL ANSWER IS $\underline{\underline{+72}}$

2. (§ 10.6, p.456 # 2). EVALUATE $\iint_S \vec{F}\cdot\vec{n}\,dS$ GIVEN $\vec{F} = \langle x^2, y^2, z^2\rangle$, S: $x+y+z=4$, $x,y,z \geq 0$

USING PARAMETRIZATION:



$x+y+z=4 \Rightarrow z = f(x,y) = 4-x-y$

LET $u=x$, $v=y \Rightarrow z = 4-u-v$

$\therefore \vec{r}(u,v) = \langle u, v, 4-u-v\rangle$

$$\Rightarrow \iint_S \vec{F}\cdot\vec{n}\,dS = \iint_{R_{uv}} \vec{F}(\vec{r})\cdot\vec{N}(u,v)\,du\,dv$$

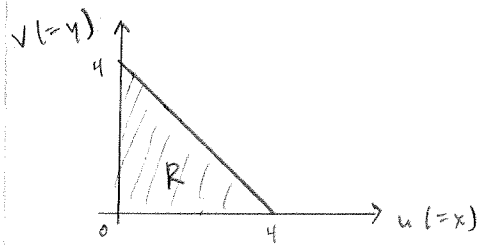
$$\vec{F}(\vec{r}(u,v)) = \langle u^2, v^2, (4-u-v)^2\rangle$$

$$\vec{N}(u,v) = \vec{r}_u \times \vec{r}_v = \langle 1, 0, -1\rangle \times \langle 0, 1, -1\rangle$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \hat{i}\begin{vmatrix} 0 & -1 \\ 1 & -1 \end{vmatrix} + \hat{j}\begin{vmatrix} 1 & -1 \\ 0 & -1 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$= \langle 1, 1, 1\rangle$$

LIMITS: NOTE THAT R_{uv} IS THE PROJECTION OF OUR SURFACE INTO THE uv-PLANE, WHICH IS THE SAME AS THE xy-PLANE SINCE $x=u$, $y=v$



$0 \leq v \leq 4$

$0 \leq u \leq 4-v$

$$\Rightarrow \int_0^4\int_0^{4-v} \langle u^2, v^2, (4-u-v)^2\rangle \cdot \langle 1,1,1\rangle\,du\,dv$$

$$= \int_0^4\int_0^{4-v} \left(u^2 + v^2 + (4-u-v)^2\right)\,du\,dv$$

$$= \int_0^4 \left[\tfrac{1}{3}u^3 + uv^2 - \tfrac{1}{3}(4-u-v)^3\right]_{u=0}^{u=4-v} dv$$

$$= \int_0^4 \left[\tfrac{1}{3}(4-v)^3 + (4-v)v^2 - \tfrac{1}{3}(4-\cancelto{0}{4+v-v})^3 + \tfrac{1}{3}(4-v)^3\right] dv$$

$$= \int_0^4 \left(\tfrac{2}{3}(4-v)^3 + 4v^2 - v^3\right) dv = -\tfrac{1}{6}(4-v)^4 + \tfrac{4}{3}v^3 - \tfrac{1}{4}v^4 \Big|_{v=0}^{v=4}$$

$$= -\tfrac{1}{6}(0)^4 + \tfrac{4}{3}(4)^3 - \tfrac{1}{4}(4)^4 + \tfrac{1}{6}(4)^4$$

$$= \underline{\underline{64}}$$