EXAMPLE PROBLEMS, LESSON 28 (GAUSS' DIVERGENCE THEOREM)

1. VERIFY GAUSS' DIVERGENCE THEOREM GIVEN $\vec{F} = \langle 7x, 0, -z \rangle$ OVER THE SPHERE $x^2+y^2+z^2=4$

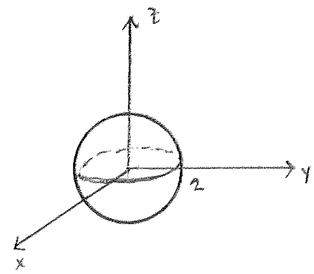
GAUSS' DIVERGENCE THM: $\iint_S \vec{F}\cdot\vec{n}\,ds = \iiint_T \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dv$

WHERE S IS THE BOUNDARY SURFACE OF THE CLOSED, BOUNDED REGION IN SPACE T AND $\vec{\nabla}\cdot\vec{F} = \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)$

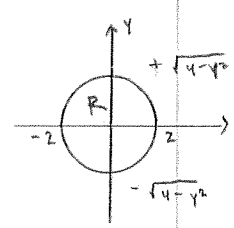
$\boxed{\iint_S \vec{F}\cdot\vec{n}\,ds}$

$S: \quad z = \pm\sqrt{4-x^2-y^2} = \pm f(x,y)$

TWO SURFACES (TOP AND BOTTOM HEMISPHERE)



PROJECTION ONTO XY-PLANE



NOTE THAT BOTH THE TOP AND BOTTOM HEMISPHERES PROJECT ONTO THE XY-PLANE

SINCE TWO SURFACES EACH HAVING A PROJECTION INTO THE SHARED REGION R, WE HAVE TO SUM TWO SURFACE INTEGRALS

$\Rightarrow \iint_S \vec{F}\cdot\vec{n}\,ds = \underbrace{\iint_R \vec{F}\cdot\langle -f_x, -f_y, 1\rangle\,dx\,dy}_{\text{TOP HEMISPHERE}} + \underbrace{\iint_R \vec{F}\cdot\langle -f_x, -f_y, 1\rangle\,dx\,dy}_{\text{BOTTOM HEMISPHERE}}$

$z = f(x,y) = \sqrt{4-x^2-y^2}$

$\Rightarrow \langle -f_x, -f_y, 1\rangle = \left\langle \frac{x}{\sqrt{4-x^2-y^2}}, \frac{y}{\sqrt{4-x^2-y^2}}, 1\right\rangle$

$\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{+\sqrt{4-y^2}} \langle 7x, 0, -\sqrt{4-x^2-y^2}\rangle\cdot\left\langle \frac{x}{\sqrt{4-x^2-y^2}}, \frac{y}{\sqrt{4-x^2-y^2}}, 1\right\rangle dx\,dy$

$= \int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{+\sqrt{4-y^2}} \left(\frac{7x^2}{\sqrt{4-x^2-y^2}} - \sqrt{4-x^2-y^2}\right) dx\,dy$

$= 32\pi$ (Mathematica)

$z = f(x,y) = -\sqrt{4-x^2-y^2}$

$\Rightarrow \langle -f_x, -f_y, 1\rangle = \left\langle \frac{-x}{\sqrt{4-x^2-y^2}}, \frac{-y}{\sqrt{4-x^2-y^2}}, 1\right\rangle$

$\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{+\sqrt{4-y^2}} \langle 7x, 0, \sqrt{4-x^2-y^2}\rangle\cdot\left\langle \frac{-x}{\sqrt{4-x^2-y^2}}, \frac{-y}{\sqrt{4-x^2-y^2}}, 1\right\rangle dx\,dy$

$= \int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{+\sqrt{4-y^2}} \left(\frac{-7x^2}{\sqrt{4-x^2-y^2}} + \sqrt{4-x^2-y^2}\right) dx\,dy$

$= 32\pi$

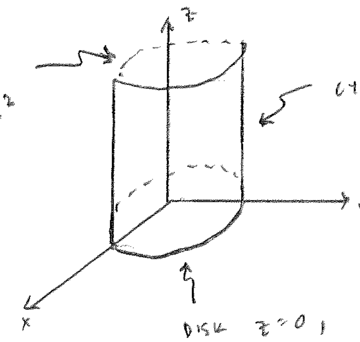
SUMMING YIELDS $\iint_S \vec{F}\cdot\vec{n}\,ds = \underline{\underline{64\pi}}$

$$\boxed{\iiint_T \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dV}$$

$$\vec{F} = \langle 7x, 0, -z \rangle \Rightarrow \vec{\nabla} \cdot \vec{F} = \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) = 7 - 1 = 6$$

$$\Rightarrow \iiint_T 6\, dV = 6 \underbrace{\iiint_T dV}_{\text{VOLUME OF THE SPHERE}} = 6\left(\frac{4}{3}\pi 2^3\right) = \underline{\underline{64\pi}}$$

2. EVALUATE $\oiint_S \vec{F} \cdot \vec{n}\, ds$ GIVEN $\vec{F} = \langle x^3, x^2 y, x^2 z \rangle$ AND S IS THE CLOSED SURFACE OF $x^2+y^2 = a^2$, $0 \le z \le b$ AND THE DISKS $z=0$, $z=b$ $(x^2+y^2 \le a^2)$



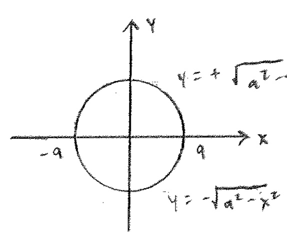
IF WE WERE TO USE SURFACE INTEGRALS, EVALUATION OF $\iint_S \vec{F} \cdot \vec{n}\, ds$ WOULD REQUIRE <u>AT LEAST</u> 3 DIFFERENT SURFACES

$\Rightarrow$ LET'S USE GAUSS' DIVERGENCE THEOREM (WILL ONLY REQUIRE ONE TRIPLE INTEGRAL)

$$\vec{F} = \langle x^3, x^2 y, x^2 z \rangle \Rightarrow \vec{\nabla} \cdot \vec{F} = 3x^2 + x^2 + x^2 = 5x^2$$

$$\Rightarrow \iiint_T \vec{\nabla} \cdot \vec{F}\, dV = \iiint_T 5x^2\, dV$$

LIMITS ? WE HAVE $0 \le z \le b$; X AND Y ARE RELATED BY $x^2+y^2 = a^2$, SO LET'S LOOK @ PROJECTION IN XY- PLANE :



$$-a \le x \le a$$
$$-\sqrt{a^2-x^2} \le y \le \sqrt{a^2-x^2}$$

$$\Rightarrow \iiint_T 5x^2\, dV = \int_0^b \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 5x^2\, dy\, dx\, dz$$
$$= \int_0^b \int_{-a}^{a} \left[5x^2 y \right]_{y=-\sqrt{a^2-x^2}}^{y=\sqrt{a^2-x^2}} dx\, dz$$
$$= \int_0^b \int_{-a}^{a} 5x^2 \left(2\sqrt{a^2-x^2} \right) dx\, dz = \underline{\underline{\frac{5}{4} a^2 b \pi}} \quad \text{(Mathematica)}$$