MA 364 Lesson #3 (Exact ODEs, Integrating Factors)

EXAMPLE PROBLEMS

① Solve $(y\cos x + 2xe^y) + (\sin x + x^2e^y - 1)\,y' = 0$

NOTE: $y' = \frac{dy}{dx}$

$$\Rightarrow \underbrace{(y\cos x + 2xe^y)}_{M(x,y)}\,dx + \underbrace{(\sin x + x^2e^y - 1)}_{N(x,y)}\,dy = 0$$

CHECK FOR EXACTNESS: $M_y = \cos x + 2xe^y$ ✓

$N_x = \cos x + 2xe^y$

$$u(x,y) = \int M(x,y)\,dx + k(y) = \int (y\cos x + 2xe^y)\,dx + k(y)$$

$$= y\sin x + x^2e^y + k(y)$$

WE KNOW (BY DEFINITION) THAT $u_y = N(x,y)$

$$\Rightarrow u_y = \sin x + x^2e^y + k'(y) = N(x,y) = \sin x + x^2e^y - 1$$

COMPARING TERM-BY-TERM WE SEE THAT $k'(y) = -1 \Rightarrow k(y) = -y + C$

$\therefore$ SOLUTIONS DEFINED IMPLICITLY BY $u(x,y) = y\sin x + x^2e^y - y + C$

ANS

(2) Solve $(3xy + y^2) + (x^2 + xy)\, y' = 0$ using the integrating factor $F(x,y) = x$

$$\Rightarrow \underbrace{(3xy + y^2)}_{M(x,y)}\, dx + \underbrace{(x^2 + xy)}_{N(x,y)}\, dy = 0$$

CHECK FOR EXACTNESS: $M_y = 3x + 2y$

$N_x = 2x + y$ NOT AN EXACT ODE

MULTIPLY THROUGH BY THE INTEGRATING FACTOR $F(x,y) = x$ AND RE-CHECK FOR EXACTNESS:

$$\Rightarrow x(3xy + y^2)\, dx + x(x^2 + xy)\, dy = 0$$

$$M_y = 3x^2 + 2xy$$

$$N_x = 3x^2 + 2xy \quad \checkmark$$

$u(x,y) = \int N(x,y)\, dy + \ell(x)$ [NOTE: WE COULD JUST AS EASILY CALCULATE $u(x,y) = \int M(x,y)\, dx + k(y)$]

$$= \int (x^3 + x^2 y)\, dy + \ell(x)$$

$$= x^3 y + \tfrac{1}{2} x^2 y^2 + \ell(x)$$

WE KNOW (BY DEFINITION) THAT $u_x = M(x,y)$

$$\Rightarrow u_x = 3x^2 y + x y^2 + \ell'(x) = M(x,y) = 3x^2 y + x y^2$$

COMPARING TERM-BY-TERM WE SEE THAT $\ell'(x) = 0 \Rightarrow \ell(x) = C$

$\therefore$ SOLUTIONS DEFINED IMPLICITLY BY $u(x,y) = x^3 y + \tfrac{1}{2} x^2 y^2 + C$

ANS