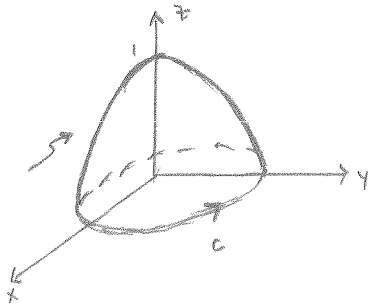
EXAMPLE PROBLEMS, MA 264 LESSON 30 (STOKES' THEOREM)

1. GIVEN $\vec{F} = \langle y, z, x \rangle$ AND S THE PARABOLOID $z = 1 - (x^2 + y^2)$, $z \geq 0$, DETERMINE $\iint_S (\vec{\nabla} \times \vec{F}) \cdot \vec{n}\, dS$

SURFACE



TWO WAYS TO SOLVE ... WE CAN ACTUALLY TAKE THE SURFACE INTEGRAL OF $\vec{\nabla} \times \vec{F}$ OVER THE PARABOLOID OR USE STOKES' THEOREM TO EVALUATE A LINE INTEGRAL OF $\vec{F}$ OVER THE ASSOCIATED BOUNDARY CURVE ...

WHICH IS EASIER?

NOTE DIRECTION OF C → "LEFT SIDE" PRINCIPLE

WHAT IS C? $z = 1 - (x^2 + y^2)$, $z \geq 0$ ⇒ IF $z = 0$, WE HAVE THE "BASE" OF THE SURFACE

$$0 = 1 - x^2 + y^2$$

$$\Rightarrow x^2 + y^2 = 1$$ SO C IS THE UNIT CIRCLE IN THE xy-PLANE

SO, C IS A RELATIVELY SIMPLE CURVE TO PARAMETERIZE AND $\vec{F}$ IS A FAIRLY SIMPLE VECTOR FIELD, SO LET'S TRY THE LINE INTEGRAL:

$$C: \vec{r}(t) = \langle \underset{x}{\cos t}, \underset{y}{\sin t}, \underset{z}{0} \rangle, \quad \underset{t_0}{0} \leq t \leq \underset{t_f}{2\pi}$$

$$\vec{r}\,'(t) = \langle -\sin t, \cos t, 0 \rangle, \quad \vec{F}(\vec{r}) = \langle \sin t, 0, \cos t \rangle$$

$$\Rightarrow \iint_S (\vec{\nabla} \times \vec{F}) \cdot \vec{n}\, dS = \int_{t_0}^{t_f} \vec{F}(\vec{r}) \cdot \vec{r}\,'(t)\, dt = \int_0^{2\pi} \langle \sin t, 0, \cos t \rangle \cdot \langle -\sin t, \cos t, 0 \rangle\, dt$$

$$= \int_0^{2\pi} -\sin^2 t\, dt$$

$$= -\left(\tfrac{1}{2}t - \tfrac{1}{4}\sin 2t\right)\Big|_0^{2\pi}$$ FORMULA IN FE MANUAL!

$$= \underline{\underline{-\pi}}$$

2. GIVEN $\vec{F} = \langle y, xz^3, -zy^3 \rangle$, $C: x^2+y^2=4$, $z=-3$, EVALUATE $\oint_C \vec{F} \cdot d\vec{r}$

$\boxed{\text{AS A LINE INTEGRAL}}$: $C: \vec{r}(t) = \langle \underset{x}{2\cos t}, \underset{y}{2\sin t}, \underset{z}{-3} \rangle$, $\underset{t_0}{0} \le t \le \underset{t_f}{2\pi}$

$$\vec{r}\,'(t) = \langle -2\sin t, 2\cos t, 0 \rangle$$

$$\vec{F}(\vec{r}) = \langle 2\sin t, -27(2\cos t), 3(2\sin t)^3 \rangle$$

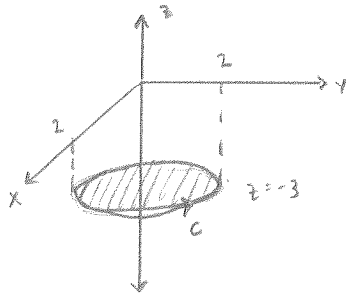
$$\Rightarrow \oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \langle 2\sin t, -54\cos t, 24\sin^3 t \rangle \cdot \langle -2\sin t, 2\cos t, 0 \rangle \, dt$$

$$= \int_0^{2\pi} -4\sin^2 t - 108\cos^2 t \, dt$$

$$= -4\left(\tfrac{1}{2}t - \tfrac{1}{4}\sin 2t\right)\Big|_0^{2\pi} - 108\left(\tfrac{1}{2}t + \tfrac{1}{4}\sin 2t\right)\Big|_0^{2\pi}$$ FORMULA IN FE MANUAL

$$= -4\pi - 108\pi = \underline{\underline{-112\pi}}$$

$\boxed{\text{USING STOKES' THEOREM}}$: NOTE THAT THERE IS NO SURFACE SPECIFIED IN THIS PROBLEM ... PICK <u>ANY</u> SURFACE BOUNDED BY THE CURVE $x^2+y^2=4$ SUCH THAT $z=-3$



THIS DISK IS THE MOST CONVENIENT CHOICE FOR A SURFACE $\longrightarrow$ $z=-3$, $x^2+y^2 \le 2$

BY STOKES' THEOREM, $\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\vec{\nabla} \times \vec{F}) \cdot \vec{n} \, dS$

WHAT IS $\vec{n}$? BY INSPECTION, $\vec{n} = \langle 0, 0, 1 \rangle$

MATHEMATICALLY, $\vec{n} = \nabla g / |\nabla g|$ WHERE $g(x,y,z) = z+3 = 0$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & xz^3 & -zy^3 \end{vmatrix} = \hat{i}\left(\frac{\partial}{\partial y}(-zy^3) - \frac{\partial}{\partial z}(xz^3)\right) - \hat{j}\left(\frac{\partial}{\partial x}(-zy^3) - \frac{\partial}{\partial z} y\right) + \hat{k}\left(\frac{\partial}{\partial x}(xz^3) - \frac{\partial}{\partial y} y\right)$$

$$= \hat{i}\left(-3zy^2 - 3xz^2\right) - \hat{j}(0-0) + \hat{k}\left(z^3 - 1\right)$$

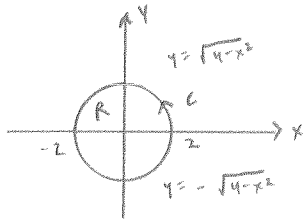
$$= \langle -3zy^2 - 3xz^2, 0, z^3 - 1 \rangle$$

BUT, WE HAVE $z=-3 \Rightarrow \vec{\nabla} \times \vec{F} = \langle 9y^2 - 27x, 0, -28 \rangle$

$$\Rightarrow \iint_S (\vec{\nabla} \times \vec{F}) \cdot \vec{n}\, dS = \iint_S \langle 9y^2 - 27x, 0, -28 \rangle \cdot \langle 0,0,1 \rangle\, dS$$

$$= \iint_S -28\, dS$$

SINCE WE HAVE $z = -3 = f(x,y)$, LET'S LOOK @ PROJECTION OF OUR SURFACE IN THE xy-PLANE:



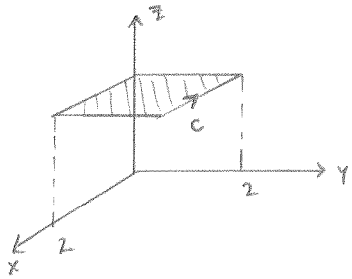
$$\Rightarrow \iint_S -28\, dS = \iint_R -28\, dA$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} -28\, dy\, dx$$

EASIER IN POLAR COORDINATES:

$$\int_0^{2\pi} \int_0^2 -28\, r\, dr\, d\theta = \underline{\underline{-112\pi}}$$

3. (§10.9, #4). FIND $\iint_S (\vec{\nabla} \times \vec{F}) \cdot \vec{n}\, dS$ GIVEN $\vec{F} = \langle 3\cos y, \cosh z, x \rangle$ AND S: SQUARE $0 \le x \le 2$, $0 \le y \le 2$, $z = 4$



ASSUME CCW ORIENTATION ON C

NOTE $\oint_C \vec{F} \cdot d\vec{r}$ PERTAINS FOR LINE INTEGRALS, SO LET'S STICK WITH $\iint_S (\vec{\nabla} \times \vec{F}) \cdot \vec{n}\, dS$

$\vec{n} = \langle 0,0,1 \rangle$ BY INSPECTION (MATHEMATICALLY, $\vec{n} = \nabla g / |\nabla g|$ WHERE $g(x,y,z) = z - 4 = 0$)

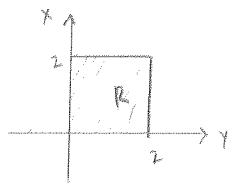
$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3\cos y & \cosh z & x \end{vmatrix} = \hat{i}\left(\frac{\partial}{\partial y} x - \frac{\partial}{\partial z} \cosh z\right) - \hat{j}\left(\frac{\partial}{\partial x} x - \frac{\partial}{\partial z} 3\cos y\right) + \hat{k}\left(\frac{\partial}{\partial x} \cosh z - \frac{\partial}{\partial y} 3\cos y\right)$$

$$= \hat{i}(0 - \sinh z) - \hat{j}(1 - 0) + \hat{k}(-(-3\sin y))$$

$$= \langle -\sinh z, -1, 3\sin y \rangle$$

$$= \langle -\sinh 4, -1, 3\sin y \rangle \quad \text{SINCE } z = 4$$

SINCE $z = 4 = f(x,y)$, PROJECT INTO xy-PLANE:



$0 \le y \le 2$

$0 \le x \le 2$

$$\Rightarrow \iint_S (\vec{\nabla} \times \vec{F}) \cdot \vec{n}\, dS = \int_0^2 \int_0^2 \langle -\sinh 4, -1, 3\sin y \rangle \cdot \langle 0,0,1 \rangle\, dx\, dy$$

$$= \int_0^2 \int_0^2 3\sin y\, dx\, dy = -(2)\, 3\cos y \Big|_0^2 = \underline{\underline{-6(\cos 2 - 1)}}$$

ANS