Lesson 4 (Homogeneous 2nd Order ODEs) Example Problems

① Verify by substitution that the given functions form a basis. Solve the IVP.

$$x^2y'' + xy' - 4y = 0, \quad x^2, \; x^{-2}, \quad y(1) = 11, \; y'(1) = -6$$

Let $y_1 = x^2$ and $y_2 = x^{-2}$. In order to establish these as a basis, we must show that y_1 and y_2 are NOT proportional and that they each individually solve the ODE

$$\frac{y_1}{y_2} = \frac{x^2}{x^{-2}} = x^4 \neq \text{constant} \Rightarrow \text{NOT proportional}$$

$$y_1 = x^2, \; y_1' = 2x, \; y_1'' = 2 \Rightarrow x^2(2) + x(2x) - 4(x^2) = 2x^2 + 2x^2 - 4x^2 = 0 \checkmark$$

$$y_2 = x^{-2}, \; y_2' = -2x^{-3}, \; y_2'' = 6x^{-4} \Rightarrow x^2(6x^{-4}) + x(-2x^{-3}) - 4(x^{-2})$$
$$= 6x^{-2} - 2x^{-2} - 4x^{-2} = 0 \checkmark$$

Since y_1 and y_2 constitute a basis, we have by the Superposition Principle that the general solution is $y = c_1y_1 + c_2y_2 = c_1x^2 + c_2x^{-2}$

Now apply the initial conditions to solve for c_1, c_2

$$y = c_1x^2 + c_2x^{-2}, \quad y' = 2c_1x - 2c_2x^{-3}$$

$$\Rightarrow y(1) = c_1 + c_2 = 11 \Rightarrow c_2 = 11 - c_1$$

$$y'(1) = 2c_1 - 2c_2 = -6$$
$$2c_1 - 2(11 - c_1) = -6 \Rightarrow 4c_1 - 22 = -6 \quad \text{or} \quad c_1 = 4$$
$$c_2 = 11 - 4 = 7$$

∴ Particular solution is $\underline{\underline{y = 4x^2 + 7x^{-2}}}$

Ans

② Find the particular solution to the given IVP:

a. $2y'' + 5y' + 3y = 0$, $y(0) = 3$, $y'(0) = -4$

b. $y'' + 16y = 0$, $y(\frac{\pi}{4}) = -3$, $y'(\frac{\pi}{4}) = 4$

c. $y'' + 12y' + 36y = 0$, $y(0) = 1$, $y'(0) = 1$

a. $2y'' + 5y' + 3y = 0$, $y(0) = 3$, $y'(0) = -4$

CHARACTERISTIC EQN: $2\lambda^2 + 5\lambda + 3 = 0 \Rightarrow \lambda_1, \lambda_2 = \dfrac{-5 \pm \sqrt{5^2 - 4(2)(3)}}{2(2)}$

$= -\frac{3}{2}, -1$

$\Rightarrow$ GENERAL SOLUTION IS $y = c_1 e^{-\frac{3}{2}x} + c_2 e^{-x}$

$y' = -\frac{3}{2} c_1 e^{-\frac{3}{2}x} - c_2 e^{-x}$

APPLY INITIAL CONDITIONS TO DETERMINE c_1 AND c_2

$y(0) = c_1 + c_2 = 3 \Rightarrow c_2 = 3 - c_1$

$y'(0) = -\frac{3}{2} c_1 - c_2 = -4 \Rightarrow -\frac{3}{2} c_1 - (3 - c_1) = -4$

$-\frac{1}{2} c_1 = -1 \Rightarrow c_1 = 2$

$c_2 = 3 - 2 = 1$

$\therefore y(x) = 2e^{-\frac{3}{2}x} + e^{-x}$

ANS

b. $y'' + 16y = 0, \quad y(\frac{\pi}{4}) = -3, \; y'(\frac{\pi}{4}) = 4$

CHARACTERISTIC EQN: $\lambda^2 + 16 = 0 \Rightarrow \lambda = \pm 4i$

$\Rightarrow$ GENERAL SOLUTION IS $y = e^{0 \cdot x}(c_1 \cos 4x + c_2 \sin 4x)$

$$= c_1 \cos 4x + c_2 \sin 4x$$

APPLY INITIAL CONDITIONS TO DETERMINE c_1 AND c_2

$$y' = -4c_1 \sin 4x + 4c_2 \cos 4x$$

$$y(\tfrac{\pi}{4}) = c_1 \cos 4(\tfrac{\pi}{4}) + c_2 \sin 4(\tfrac{\pi}{4})$$

$$= c_1 \underbrace{\cos \pi}_{-1} + c_2 \underbrace{\sin \pi}_{0} = -3$$

$$\Rightarrow -c_1 = -3 \quad \text{OR} \quad c_1 = 3$$

$$y'(\tfrac{\pi}{4}) = -4c_1 \sin 4(\tfrac{\pi}{4}) + 4c_2 \cos 4(\tfrac{\pi}{4})$$

$$= -4(3) \underbrace{\sin \pi}_{0} + 4c_2 \underbrace{\cos \pi}_{-1} = 4$$

$$\Rightarrow -4c_2 = 4 \quad \text{OR} \quad c_2 = -1$$

$\therefore \; y(x) = 3\cos 4x - \sin 4x$

ANS

c. $y'' + 12y' + 36y = 0$, $y(0) = 1$, $y'(0) = 1$

CHARACTERISTIC EQN: $\lambda^2 + 12\lambda + 36 = 0 \Rightarrow (\lambda+6)^2 = 0$

$\lambda = -6$

$\Rightarrow$ GENERAL SOLUTION IS $y = c_1 e^{-6x} + c_2 x e^{-6x}$

APPLY INITIAL CONDITIONS TO DETERMINE c_1 AND c_2

$$y' = -6c_1 e^{-6x} + c_2 e^{-6x} - 6c_2 x e^{-6x}$$

$$y(0) = c_1 = 1$$

$$y'(0) = -6c_1 + c_2 = 1$$

$$-6(1) + c_2 = 1 \Rightarrow c_2 = 7$$

$\therefore$ $y(x) = e^{-6x} + 7xe^{-6x}$

ANS