Lesson 5 Example Problems (Non-Homogeneous 2nd order ODEs, Method of Undetermined Coefficients)

① Solve $y'' + 3y' + 2y = 30e^{2x}$

$y = y_h + y_p$

$\boxed{y_h}$ $y'' + 3y' + 2y = 0 \Rightarrow$ CHARACTERISTIC EQN. $\lambda^2 + 3\lambda + 2 = 0$

$(\lambda+2)(\lambda+1) = 0$

$\Rightarrow \lambda = -2, -1$

$\therefore y_h = c_1 e^{-2x} + c_2 e^{-x}$

$\boxed{y_p}$ $y_p = Ce^{2x}$

$y_p' = 2Ce^{2x}$, $y_p'' = 4Ce^{2x}$

$\Rightarrow y_p'' + 3y_p' + 2y_p = 4Ce^{2x} + 3(2Ce^{2x}) + 2(Ce^{2x}) = 30e^{2x}$

$4Ce^{2x} + 6Ce^{2x} + 2Ce^{2x} = 30e^{2x}$

$12Ce^{2x} = 30e^{2x} \Rightarrow C = \frac{30}{12} = \frac{5}{2}$

$\therefore y = y_h + y_p$

$= c_1 e^{-2x} + c_2 e^{-x} + \underline{\frac{5}{2}e^{2x}}$

ANS

NOTE: IF WE HAD ICs, WE WOULD APPLY THEM NOW TO SOLVE FOR c_1, c_2

② Solve $y'' + 4y' + 3.75y = 109 \cos 5x$

$y = y_h + y_p$

$\boxed{y_h}$ $y'' + 4y' + 3.75y = 0 \Rightarrow$ CHARACTERISTIC EQN. $\lambda^2 + 4\lambda + 3.75 = 0$

$$\lambda = \frac{-4 \pm \sqrt{4^2 - 4(1)(3.75)}}{2(1)}$$

$$= -\frac{5}{2}, -\frac{3}{2}$$

$$\Rightarrow y_h = c_1 e^{-5/2\,x} + c_2 e^{-3/2\,x}$$

$\boxed{y_p}$ LET $y_p = k \cos 5x + M \sin 5x$

$y_p' = -5k \sin 5x + 5M \cos 5x$

$y_p'' = -25k \cos 5x - 25M \sin 5x$

$$\Rightarrow y_p'' + 4y_p' + 3.75y_p = -25k\cos 5x - 25M \sin 5x + 4(-5k \sin 5x + 5M \cos 5x) + 3.75(k \cos 5x + M \sin 5x) = 109 \cos 5x$$

$$(-25k + 20M + 3.75k)\cos 5x + (-25M - 20k + 3.75M)\sin 5x = 109 \cos 5x$$

$$\Rightarrow \left.\begin{aligned} -21.25k + 20M &= 109 \\ -21.25M - 20k &= 0 \end{aligned}\right\} \longrightarrow M = 2.56,\ k = -2.72$$

$\therefore$ $y = y_h + y_p$

$$= c_1 e^{-2.5x} + c_2 e^{-1.5x} - 2.72 \cos 5x + 2.56 \sin 5x$$

ANS

③ Solve $y'' - 2y' - 3y = 8e^{-t}$

$y = y_h + y_p$

$\boxed{y_h}$ $y'' - 2y' - 3y = 0 \Rightarrow$ CHARACTERISTIC EQN. $\lambda^2 - 2\lambda - 3 = 0$

$(\lambda - 3)(\lambda + 1) = 0$

$\Rightarrow \lambda = 3, -1$

$\therefore \; y_h = c_1 e^{-t} + c_2 e^{3t}$

$\boxed{y_p}$ WE WOULD NORMALLY SELECT $y_p = Ce^{-t}$ AS OUR CANDIDATE PARTICULAR SOLUTION, BUT THIS APPEARS AS A TERM IN THE HOMOGENEOUS SOLUTION. THIS IS KNOWN AS RESONANCE, AND THE METHOD OF UNDETERMINED COEFFICIENTS HAS TO BE MODIFIED ACCORDINGLY $\Rightarrow$ MULTIPLY THE CANDIDATE y_p BY A FACTOR OF t!

NOTE: WE WILL SOMETIMES HAVE TO MULTIPLY THE CANDIDATE y_p BY $t^2 \rightarrow$ SEE EXAMPLE #2, p. 81, § 2.7 IN KREYSZIG

$y_p = Cte^{-t} \Rightarrow y_p' = Ce^{-t} - Cte^{-t}$, $y_p'' = -Ce^{-t} - Ce^{-t} + Cte^{-t}$

$= -2Ce^{-t} + Cte^{-t}$

$\Rightarrow \; y_p'' - 2y_p' - 3y_p = (Cte^{-t} - 2Ce^{-t}) - 2(Ce^{-t} - Cte^{-t}) - 3Cte^{-t} = 8e^{-t}$

$\cancel{Cte^{-t}} - 2Ce^{-t} - 2Ce^{-t} + 2\cancel{Cte^{-t}} - 3\cancel{Cte^{-t}} = 8e^{-t}$

$-4Ce^{-t} = 8e^{-t} \Rightarrow C = -2$

$\therefore \; y = y_h + y_p = c_1 e^{-t} + c_2 e^{3t} - 2e^{-t}$

ANS

OTHER EXAMPLES (TEXT): EXAMPLES #1, 3, p. 80-82, § 2.7, KREYSZIG