Lesson 6 (Non homogeneous ODEs, Modeling Forced Oscillations) Example Problems:

① (p. 90, #3). Find the steady state oscillations of the mass-spring system $y'' + y' + 4.25y = 221 \cos 4.5t$

$my'' + cy' + ky = F_0 \cos \omega t \Rightarrow m = 1,\ c = 1,\ k = 4.25,\ F_0 = 221,\ \omega = 4.5$

NOTE SINCE $c \neq 0$, WE HAVE A DAMPED FORCED OSCILLATION

$$\Rightarrow y_{\text{STEADY-STATE}} = y_p = a \cos \omega t + b \sin \omega t$$

$$= a \cos 4.5t + b \sin 4.5t$$

$$a = F_0 \frac{m(\omega_0^2 - \omega^2)}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2}$$

$$b = F_0 \frac{\omega c}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2}$$

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{4.25}{1}} = 2.06155$$

$$\Rightarrow a = 221 \frac{(1)(2.06155^2 - 4.5^2)}{(1)^2(2.06155^2 - 4.5^2)^2 + (4.5)^2(1)^2} = -12.8$$

$$b = 221 \frac{4.5(1)}{(1)^2(2.06155^2 - 4.5^2)^2 + (4.5)^2(1)^2} = 3.6$$

$$\therefore y_{ss} = y_p = -12.8 \cos 4.5t + 3.6 \sin 4.5t$$

ANS

(p. 90, #9)

② A Find the transient motion of the mass-spring system modeled by the ODE $y'' + 2y' + 0.75y = 13 \sin t$

NOTE THAT SINCE $c \neq 0$, WE HAVE A <u>DAMPED</u> FORCED OSCILLATION

SINCE OUR FORCING FUNCTION $13 \sin t$ IS NOT OF THE FORM $F_0 \cos \omega t$, WE CANNOT APPEAL TO FORMULAS; HOWEVER, THOSE FORMULAS SIMPLY COME FROM THE METHOD OF UNDETERMINED COEFFICIENTS, WHICH WE WILL USE HERE

ALSO, THE <u>TRANSIENT</u> SOLUTION IS $y = y_h + y_p$; FOR DAMPED OSCILLATIONS, IF WE WERE ASKED TO PROVIDE THE <u>STEADY-STATE</u> SOLUTION, WE WOULD SIMPLY HAVE TO FIND y_p SINCE $y_{ss} = y_p$

$\boxed{y_h}$ $y'' + 2y' + 0.75y = 0 \Rightarrow$ CHARACTERISTIC EQN $\lambda^2 + 2\lambda + 0.75 = 0$

$$\lambda = \frac{-2 \pm \sqrt{2^2 - 4(1)(0.75)}}{2(1)}$$

$$= -\tfrac{1}{2}, -\tfrac{3}{2}$$

$$\Rightarrow y_h = c_1 e^{-1/2 t} + c_2 e^{-3/2 t}$$

$\boxed{y_p}$ $y_p = a \cos \omega t + b \sin \omega t$ $\quad \omega = 1$ (since our forcing function is $13 \sin \underset{\omega = 1}{t}$)

$= a \cos t + b \sin t$

$y_p' = -a \sin t + b \cos t$

$y_p'' = -a \cos t - b \sin t$

$y_p'' + 2y_p' + 0.75 y_p = 13 \sin t \Rightarrow -a \cos t - b \sin t + 2(-a \sin t + b \cos t) + 0.75(a \cos t + b \sin t) = 13 \sin t$

$(-a + 2b + 0.75a) \cos t + (-b - 2a + 0.75b) \sin t = 13 \sin t$

$$\Rightarrow \left.\begin{aligned} -0.25a + 2b &= 0 \\ -0.25b - 2a &= 13 \end{aligned}\right\} \rightarrow a = -6.4,\ b = -0.8$$

$$\therefore y = y_h + y_p = c_1 e^{-1/2 t} + c_2 e^{-3/2 t} - 6.4 \cos t - 0.8 \sin t$$

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③ (p. 90, #20) MODIFIED FROM TEXT Find the motion of the mass-spring system modeled by the ODE $y'' + 25y = 99\cos 4.9t$

$my'' + cy' + ky = F_0 \cos \omega t \Rightarrow m = 1,\ c = 0,\ k = 25,\ F_0 = 99,\ \omega = 4.9$

NOTE THAT SINCE $c = 0$, WE HAVE AN UNDAMPED FORCED OSCILLATION, SO WE MUST CHECK FOR RESONANCE

$\omega = \omega_0$? $\quad \omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{25}{1}} = 5 \neq 4.9$ NO!

$\boxed{y_h}$ $y'' + 25y = 0 \Rightarrow$ CHARACTERISTIC EQN. $\lambda^2 + 25 = 0$

$\lambda = \pm 5i$

$\Rightarrow y_h = \cancel{e^{0 \cdot t}}^{\,1}(c_1 \cos 5t + c_2 \sin 5t)$

$= c_1 \cos 5t + c_2 \sin 5t$

$\boxed{y_p}$ $y_p = a \cos \omega t + b \sin \omega t$ (NO RESONANCE, ELSE WE WOULD USE THE CANDIDATE $y_p = t(a \cos \omega t + b \sin \omega t)$)

$= a \cos 4.9t + b \sin 4.9t$

SINCE UNDAMPED, NO RESONANCE, WE KNOW THAT $a = F_0\left(\frac{1}{m(\omega_0^2 - \omega^2)}\right)$, $b = 0$

$\Rightarrow a = 99\left(\frac{1}{(1)(5^2 - 4.9^2)}\right) = 99(1.0101) = 100$

$\therefore\ y = y_h + y_p = c_1 \cos 5t + c_2 \sin 5t + \underline{\underline{100 \cos 4.9t}}$

ANS