EXAMPLE PROBLEMS, LESSONS 19 + 20

1. FIND THE PARAMETRIC REPRESENTATION OF THE CIRCLE OF RADIUS 3 CENTERED @ (1,2)

EQN. OF CIRCLE : $(x-1)^2 + (y-2)^2 = 3^2$

NOTE THAT IF WE CAN PERFORM ALGEBRA TO REWRITE THIS EQUATION IN THE FORM $(\quad)^2 + (\quad)^2 = 1$, WE CAN ASSOCIATE THE W/ THE IDENTITY $\cos^2 t + \sin^2 t = 1$

$$\Rightarrow \left(\frac{x-1}{3}\right)^2 + \left(\frac{y-2}{3}\right)^2 = 1$$

$$\Rightarrow \left(\frac{x-1}{3}\right)^2 = \cos^2 t \quad \text{AND} \quad \left(\frac{y-2}{3}\right)^2 = \sin^2 t$$

$$\Rightarrow \frac{x-1}{3} = \cos t \quad \text{OR} \quad x = 1 + 3\cos t$$

$$\frac{y-2}{3} = \sin t \quad \text{OR} \quad y = 2 + 3\sin t$$

$$\therefore \quad x(t) = 1 + 3\cos t \quad \text{AND} \quad y(t) = 2 + 3\sin t$$

ANS

2. GIVEN $\vec{r}(t) = \langle 3\cos t, 3\sin t, 4t \rangle$ AND $P: (3, 0, 8\pi)$, FIND A TANGENT VECTOR $\vec{r}\,'(t)$, A UNIT TANGENT VECTOR $\vec{u}\,'(t)$, AND THE TANGENT OF THE CURVE AT P

$$\vec{r}\,'(t) = \langle -3\sin t, 3\cos t, 4 \rangle$$

$$\vec{u}\,'(t) = \frac{1}{|\vec{r}\,'(t)|}\,\vec{r}\,'(t) = \frac{1}{\sqrt{(-3\sin t)^2 + (3\cos t)^2 + 4^2}} \langle -3\sin t, 3\cos t, 4 \rangle$$

$$= \frac{1}{\sqrt{9(\sin^2 t + \cos^2 t) + 16}} \langle -3\sin t, 3\cos t, 4 \rangle$$

$$= \frac{1}{\sqrt{25}} \langle -3\sin t, 3\cos t, 4 \rangle = \langle -\tfrac{3}{5}\sin t, \tfrac{3}{5}\cos t, \tfrac{4}{5} \rangle$$

NOTE THAT $P: (3, 0, 8\pi)$ CORRESPONDS TO $t = 2\pi$ BECAUSE $\vec{r}(2\pi) = \langle 3\cos 2\pi, 3\sin 2\pi, 4(2\pi) \rangle = \langle 3, 0, 8\pi \rangle$

$\Rightarrow \quad \vec{r}\,'(2\pi) = \langle -3\sin 2\pi,\ 3\cos 2\pi,\ 4 \rangle = \langle 0, 3, 4 \rangle$

$\vec{u}\,'(2\pi) = \langle 0, \frac{3}{5}, \frac{4}{5} \rangle$

THE TANGENT LINE IS PARAMETRICALLY REPRESENTED BY

$$\vec{q}(w) = \vec{r}(2\pi) + w\,\vec{r}\,'(2\pi) \qquad [\text{EQN } \#9,\ p.\ 392]$$

$$= \langle 3, 0, 8\pi \rangle + w\,\langle 0, 3, 4 \rangle$$

$$= \langle 3, 3w, 8\pi + 4w \rangle$$

ANS

3. FIND THE ARCLENGTH OF $\vec{r}(t) = \langle 2\cos t, 2\sin t, 6t \rangle$ FROM $(2,0,0)$ TO $(2, 0, 24\pi)$

NOTE THAT $(2,0,0)$ CORRESPONDS TO $t=0$ AND $(2,0,24\pi)$ TO $t = 4\pi$

$x(t) = 2\cos t \Rightarrow x'(t) = -2\sin t$
$y(t) = 2\sin t \Rightarrow y'(t) = 2\cos t$
$z(t) = 6t \Rightarrow z'(t) = 6$

$$\Rightarrow \quad \ell = \int_0^{4\pi} \sqrt{(-2\sin t)^2 + (2\cos t)^2 + 6^2}\ dt$$

$$= \int_0^{4\pi} \sqrt{4(\sin^2 t + \cos^2 t) + 36}\ dt$$

$$= \int_0^{4\pi} \sqrt{40}\ dt = t\sqrt{40}\Big|_{t=0}^{t=4\pi} = 4\pi\sqrt{40}$$

$$= 8\pi\sqrt{10} \quad \text{UNITS OF LENGTH}$$

ANS

4. FIND THE VELOCITY, SPEED, TANGENTIAL, AND NORMAL ACCELERATIONS FOR THE MOTION $\vec{r}(t) = \langle t, t^2, 0 \rangle$

$\vec{v}(t) = \vec{r}\,'(t) = \langle 1, 2t, 0 \rangle$

$|\vec{v}\,'(t)| = \text{SPEED} = \sqrt{1^2 + (2t)^2 + 0^2} = \sqrt{1 + 4t^2}$

$\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t) = \langle 0, 2, 0 \rangle$

$$\vec{a}_{tan} = \frac{\vec{a}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\,\vec{v} = \frac{\langle 0,2,0\rangle \cdot \langle 1,2t,0\rangle}{\langle 1,2t,0\rangle \cdot \langle 1,2t,0\rangle}\langle 1,2t,0\rangle = \frac{4t}{1+4t^2}\langle 1,2t,0\rangle$$

$$= \left\langle \frac{4t}{1+4t^2}, \frac{8t^2}{1+4t^2}, 0 \right\rangle$$

$$\vec{a}_{norm} = \vec{a} - \vec{a}_{tan}$$

$$= \langle 0, 2, 0 \rangle - \left\langle \frac{4t}{1+4t^2}, \frac{8t^2}{1+4t^2}, 0 \right\rangle$$

$$= \left\langle \frac{-4t}{1+4t^2}, 2 - \frac{8t^2}{1+4t^2}, 0 \right\rangle$$

ANS

5. FIND A NORMAL VECTOR OF THE SURFACE $x^2 + 3y^2 + z^2 = 28$ AT $P: (4, 1, 3)$

$$f(x,y,z) = x^2 + 3y^2 + z^2 = \underbrace{28}_{\text{const.}}$$

$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \langle 2x, 6y, 2z \rangle$ NORMAL VECTOR (GRADIENT)

$\nabla f(4,1,3) = \langle 2(4), 6(1), 2(3) \rangle = \langle 8, 6, 6 \rangle$ VALUE OF NORMAL VECTOR @ P

ANS

6. FIND THE DIRECTIONAL DERIVATIVE OF $f = e^x \sin y$ IN THE DIRECTION IF $\vec{a} = \langle 2, 3, 0 \rangle$ AT $P: (2, \frac{\pi}{2}, 0)$

$$D_{\vec{a}} f = \frac{1}{|\vec{a}|} \vec{a} \cdot \nabla f$$

$$|\vec{a}| = \sqrt{2^2 + 3^2 + 0^2} = \sqrt{4+9} = \sqrt{13} \Rightarrow \frac{1}{|\vec{a}|} = \frac{1}{\sqrt{13}}$$

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \langle e^x \sin y, e^x \cos y, 0 \rangle$$

$$\Rightarrow \nabla f(2, \tfrac{\pi}{2}, 0) = \langle e^2 \sin \tfrac{\pi}{2}, e^2 \cos \tfrac{\pi}{2}, 0 \rangle = \langle e^2, 0, 0 \rangle$$

$$\vec{a} \cdot \nabla f(2, \tfrac{\pi}{2}, 0) = \langle 2, 3, 0 \rangle \cdot \langle e^2, 0, 0 \rangle = 2e^2$$

$$\Rightarrow D_{\vec{a}} f = \frac{2e^2}{\sqrt{13}}$$

ANS