EXAMPLE PROBLEMS, LESSONS 21 and 22 (DIVERGENCE, CURL)

1. FIND THE DIVERGENCE OF $\vec{v} = \langle 3x^2y,\ 2z,\ 3xyz^3 \rangle$

$$\vec{\nabla} \cdot \vec{v} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle 3x^2y,\ 2z,\ 3xyz^3 \rangle$$

$$= \frac{\partial}{\partial x}(3x^2y) + \frac{\partial}{\partial y}(2z) + \frac{\partial}{\partial z}(3xyz^3)$$

$$= 6xy + 0 + 9xyz^2$$

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SO, THE DIVERGENCE OF $\vec{v}$ AT SOME ARBITRARY POINT (x, y, z) IS PROVIDED BY THE SCALAR FUNCTION $\underline{\underline{6xy + 9xyz^2}}$

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2. LET $\vec{v} = \langle 0, z^2, 0 \rangle$ BE THE VELOCITY VECTOR FIELD OF A STEADY FLUID FLOW. IS THE FLOW INCOMPRESSIBLE? IRROTATIONAL? FIND THE STREAMLINES.

INCOMPRESSIBLE? $\vec{\nabla} \cdot \vec{v} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle 0, z^2, 0 \rangle$

$$= \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(z^2) + \frac{\partial}{\partial z}(0)$$

$$= 0 \quad \checkmark$$

IRROTATIONAL? $\vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & z^2 & 0 \end{vmatrix} = \hat{i} \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & 0 \end{vmatrix} - \hat{j} \cancelto{0}{\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ 0 & 0 \end{vmatrix}} + \hat{k} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ 0 & z^2 \end{vmatrix}$

$$= \hat{i}\left(\frac{\partial}{\partial y}0 - \frac{\partial}{\partial z}z^2\right) + \hat{k}\left(\frac{\partial}{\partial x}z^2 - \frac{\partial}{\partial y}0\right)$$

$$= -2z\,\hat{i} = \langle -2z, 0, 0 \rangle \neq \vec{0}$$

$\underline{\text{NOT}}$ IRROTATIONAL!

NOTE THAT $\vec{v}(t) = \langle 0, z^2, 0 \rangle = \vec{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$

$\Rightarrow$ $\left.\begin{array}{l} x'(t) = 0 \\ y'(t) = z^2 \\ z'(t) = 0 \end{array}\right\} \longrightarrow$ $\begin{array}{l} x(t) = c_1 \\ z(t) = c_3 \end{array}$ $\quad \begin{array}{l} y'(t) = z^2 = c_3^2 \\ \Rightarrow y(t) = c_3^2 t + c_2 \end{array}$

$\therefore$ OUR STREAMLINES (THE PATHS OF PARTICLES UNDER FLOW DESCRIBED BY $\vec{v}$) IS $\vec{r}(t) = \langle c_1,\ c_3^2 t + c_2,\ c_3 \rangle$ WHERE $c_1, c_2,$ AND c_3 ARE CONSTANTS

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