Application of Divergence Theorem

Heat Flow, Diffusion Equation:

words: * Heat flows in the direction of decreasing temperature A is proportional to the gradient of temperature *

$$\vec{v} = \vec{\nabla} u(x,y,z)\,(-k)$$

→ equation

$\vec{v}$: heat flow

$\vec{\nabla} u(x,y,z)$: temp function; gradient of temp

$(-k)$: proportional constant; thermal conductivity

Now, lets find amt of heat leaving an object per unit time

↳ this becomes a surface integral:

$$\iint_S \vec{v} \cdot \vec{n}\, dS = \frac{\text{amt of heat}}{\text{unit time}}$$

$\vec{v}$: heat flow

$\vec{n}$: normal to surface

Since a single object is surrounded by a closed surface, we can use the Divergence Theorem:

$$\iint_S \vec{v} \cdot \vec{n}\, dS = \iiint_T \vec{\nabla} \cdot \vec{v}\, dV$$

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Now apply what we know about $\vec{v}$,

$$\vec{v} = -k\vec{\nabla}u$$

we get:

$$-\frac{\Delta \text{heat}}{\Delta \text{time}} = \iiint_T \vec{\nabla}\cdot(-k)\vec{\nabla}u \, dV$$

$$= -k\iiint_T \vec{\nabla}\cdot\vec{\nabla}u \, dV$$

(k: pulled out since it's const; $\vec{\nabla}\cdot\vec{\nabla}u$: Laplacian)

$$= -k\iiint_T \nabla^2 u \, dV = -\frac{\Delta \text{heat}}{\Delta \text{time}}$$

Can we use this without knowing $u(x,y,z)$? No

So, is there another technique to find $\frac{\Delta \text{heat}}{\Delta \text{time}}$? Well, what is $\frac{\Delta \text{heat}}{\Delta \text{time}}$ ~~in the~~ as we let $\Delta \text{time} \to 0$?

- $\frac{\partial \text{Heat}}{\partial t}$ } time derivative of the amount of heat in an object!

so

$$-\frac{\partial H}{\partial t} = -k\iiint_T \nabla^2 u \, dV$$

still doesn't help since don't know u or H, but can we find H in relation to u? sure!

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H = total amount of heat in object

$$H = \iiint_T \sigma \rho U \, dV$$

σ = specific heat of material (const)
ρ = density of material

so what is $\frac{\partial H}{\partial t}$?

$$\frac{\partial H}{\partial t} = \frac{\partial}{\partial t}\left(\iiint_T \sigma \rho U \, dV\right)$$

Then the time rate of decrease of H is:

$$-\frac{\partial H}{\partial t} = \frac{-\partial}{\partial t} \iiint \sigma \rho U \, dV$$

since σ & ρ are const w/respect to time

Now equating the two $\frac{\partial H}{\partial t}$s:

$$-\iiint_T \sigma \rho \frac{\partial U}{\partial t} \, dV = -k \iiint_T \nabla^2 U \, dV$$

or

$$\iiint_T \sigma \rho \frac{\partial U}{\partial t} - k \nabla^2 U \, dV = 0$$

so

$$\sigma \rho \frac{\partial U}{\partial t} - K \nabla^2 U = 0$$

or $$\frac{\partial U}{\partial t} = c^2 \nabla^2 U, \quad c^2 = \frac{k}{\sigma \rho}$$

c = thermal diffusivity

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By using the divergence equation, we were able to ~~so~~ derive the heat equation:

$$\frac{\partial u}{\partial t} = c^2 \nabla^2 u$$

We still don't know u? but one eq, one unknown

What we have is a partial differential equation which we'll learn to solve in Block 8.

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Moments of Inertia

2-D surface: $I = \iint_S \mu D^2 \, dS$ or 3-D object: $\iiint_V \mu D^2 \, dV$

↳ D is the distance function of the point (x,y,z) on/in the object from L. L is the axis w/which you are getting the moment of inertia w/respect too.

↳ μ = density function

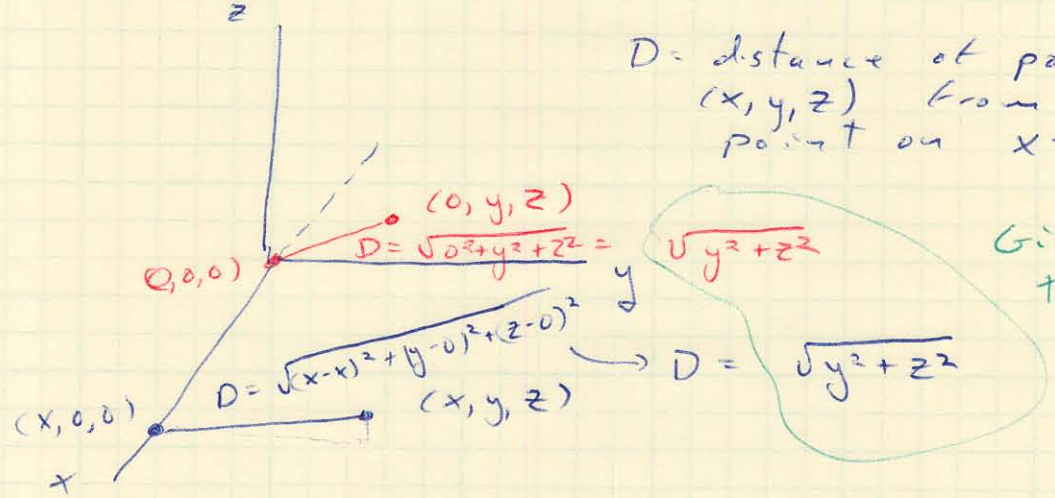
Ex: P. 9 pg 463

Given: $I_x = \iiint_T (y^2+z^2)\,dx\,dy\,dz$ of a cube, with $\mu = 1$, & $0 \le x \le a$, $0 \le y \le a$, $0 \le z \le a$

Find: I_x

What is D? Draw it out!

D = distance of point (x,y,z) from closest point on x-axis



Given in the integral but good to verify & see where it came from

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So all that is left is solving the integral:

$$\int_0^a \int_0^a \int_0^a (y^2+z^2)\,dx\,dy\,dz$$

$$= \int_0^a \int_0^a x(y^2+z^2)\Big|_0^a \,dy\,dz$$

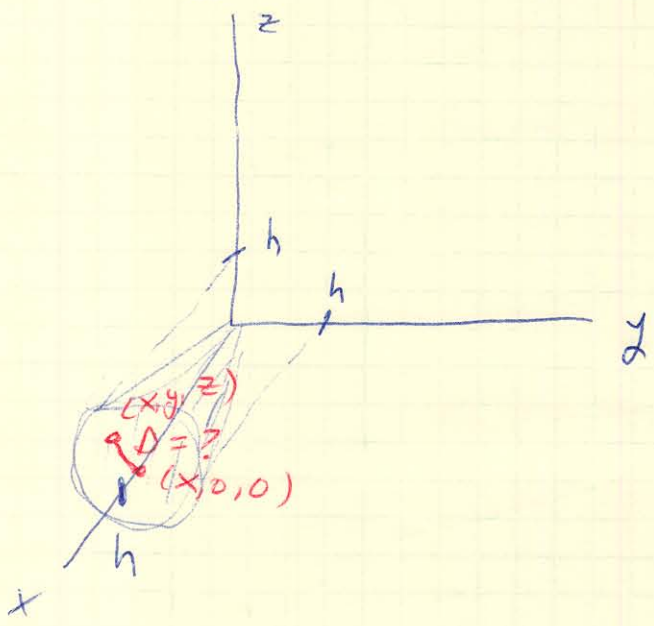
$$= \int_0^a a\left(\frac{y^3}{3} + z^2 y\right)\Big|_0^a \,dz$$

$$= \int_0^a \frac{a^4}{3} + a^2 z^2 \,dz$$

$$= \frac{a^4}{3} z + \frac{a^2 z^3}{3}\Big|_0^a = \frac{2a^5}{3}$$

Let's try another object! P. 13, pg 463

Cone:



Does D change?
No!

Cone: $y^2+z^2 \le x^2$
$0 \le x \le h$

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So $I_x = \iiint_T (y^2+z^2)\,dx\,dy\,dz$

Lets Find the limits of integration:

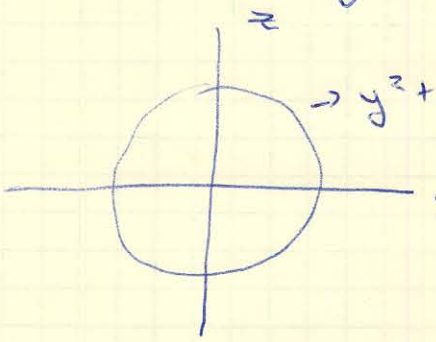
$0 \le x \le h$ → not dependent on any variable, we'll make this the last integral

$y^2+z^2 \le x^2$ → both y & z depend on x & each other.

Lets choose to integrate y first:

~~$\sqrt{x^2-z^2} \le y \le \sqrt{y^2-z^2}$~~

Look in ~~yz~~ yz plane at $x = x$.

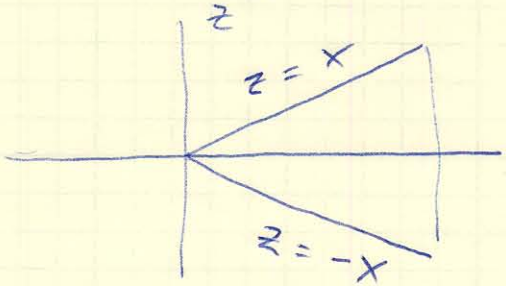


→ $y^2+z^2 = x^2$

so y limits:

$-\sqrt{x^2-z^2} \le y \le \sqrt{x^2-z^2}$

For z-limits, lets look in xz plane



so

$-x \le z \le x$

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Now set up Integral:

$$\int_0^h \int_{-x}^{x} \int_{-\sqrt{x^2-z^2}}^{\sqrt{x^2-z^2}} (y^2+z^2)\, dy\, dx\, dz$$

$$= \frac{h^5 \pi}{10}$$

Bonus! Is there an easier way, if so, how? if not support your answer

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General Theorem Info!

1. Line Integrals

$\int_C \vec{F} \cdot d\vec{r}$ → Work Problems, $\vec{F}$ = Force

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

($d\vec{r}$)

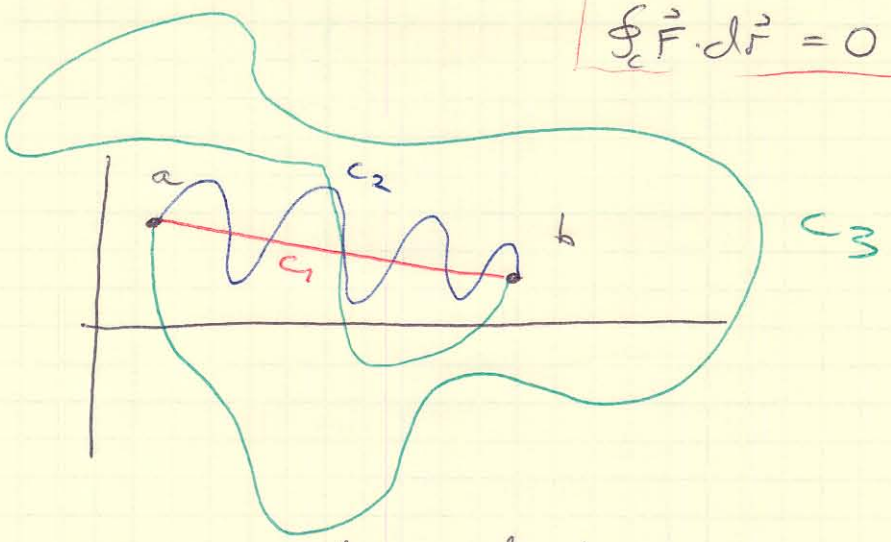
where C is parametrically represented by $\vec{r}(t)$

→ Path Independence:

if $\vec{F} = \vec{\nabla}\phi$, Then path independent, conservative, exact

$\vec{\nabla} \times \vec{F} = \vec{0}$ also

$\oint_C \vec{F} \cdot d\vec{r} = 0$



if Path independent, then:

$$\int_{C_1} \vec{F} \cdot d\vec{r}_1 = \int_{C_2} \vec{F} \cdot d\vec{r}_2 = \int_{C_3} \vec{F} \cdot d\vec{r}_3$$

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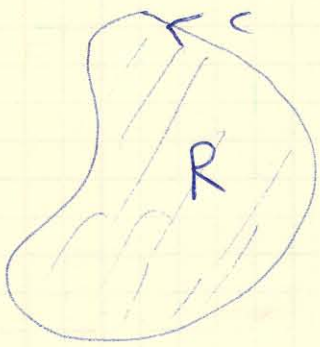
2. Green's Theorem: Line Integrals Easier

$\vec{F} = \langle F_1, F_2 \rangle$

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\,dy$$



if This = 0, then $\vec{F}$ is exact & $\iint_R = 0$, so the line integral = 0 & is Path independent

3. Surface Integrals

a. Given a surface represented as a parametric equation,

$$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$$

Calculate $\iint_S \vec{F} \cdot \vec{n}\, dS$ ⇒ Flux integral

$$\iint_S \vec{F} \cdot \vec{n}\, dS = \iint_{R_{uv}} \vec{F} \cdot \underbrace{\frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}}_{\vec{n}} \underbrace{|\vec{r}_u \times \vec{r}_v|\, du\,dv}_{dS}$$

$$= \iint_{R_{uv}} \vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\, du\,dv$$

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b. Given a surface represented as:

$$z = f(x,y)$$
or $y = F(x,z)$
or $x = F(y,z)$

then (for $z = F(x,y)$)

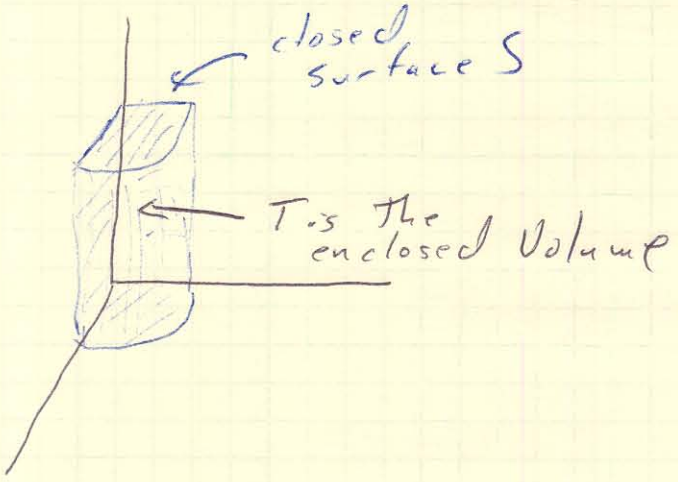
$$\iint_S \vec{F}\cdot\hat{n}\,dS = \iint_R \vec{F}\cdot \underbrace{\frac{\langle -F_x, -F_y, 1\rangle}{\sqrt{F_x^2+F_y^2+1}}}_{\vec{n}} \underbrace{\sqrt{F_x^2+F_y^2+1}\,dx\,dy}_{dS}$$

$$= \iint_R \vec{F}\cdot\langle -F_x, -F_y, 1\rangle\,dx\,dy$$

4. Divergence Theorem: Surface integrals easier

Given a closed surface S,

$$\iint_S \vec{F}\cdot\hat{n}\,dS = \iiint_T \vec{\nabla}\cdot\vec{F}\,dV$$



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