# Divergence Theorem

Theorem: (surface integral)

$$\iiint_T \operatorname{div} \vec{F}\, dV = \iint_S \vec{F} \cdot \vec{n}\, dS$$

↳ S must be orientable (what does that mean?) and closed

$$\operatorname{div} \vec{F} = \vec{\nabla} \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\iiint_T \left( \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dx\, dy\, dz$$

$$= \iint_S F_1\, dy\, dz + F_2\, dx\, dz + F_3\, dx\, dy$$

→ one of the texts versions of a surface integral

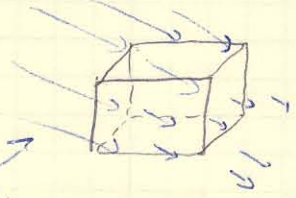
Can you see how this came from above?

$$= \iint_A \vec{F}(x, y, z(x,y)) \cdot \langle -F_x, -F_y, 1 \rangle\, dx\, dy \quad \text{if } z = f(x,y)$$

] from last set of notes

So how & why do we use this theorem:

imagine wanting to calculate the amount of material flowing through a box:



Velocity Field or Flux

} How many surface integrals would it take?

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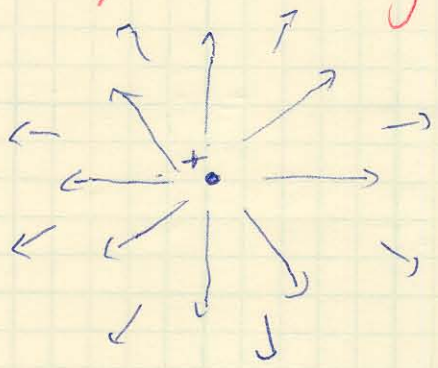
We use the divergence theorem to calculate flux/surface integrals through closed orientable surfaces.

This is extremely powerful! imagine a point charge:



How would you calculate the flux integral through any closed surface, $S$, not containing the charge?

What do we know about electric flux?

$$\underbrace{\vec{\nabla}\cdot\vec{B}}_{\text{Div of Flux}} = \underbrace{\rho}_{\text{charge dens. of}},$$

So if our closed surface has enclosed a net charge of 0, then

$$\vec{\nabla}\cdot\vec{B} = 0, \quad \&$$

$$\iiint_T \vec{\nabla}\cdot\vec{B}\, dV = 0 = \oiint_S \vec{B}\cdot\vec{n}\, dS$$

(the $\oiint$ symbolizes a closed surface integral)

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Let's look at example 1 from book:

Given: $I = \iint_S x^3\,dy\,dz + x^2y\,dz\,dx + x^2z\,dx\,dy$

This is our surface integral, but what is $\vec{F}$?

refer to setup:

$$\iint_S F_1\,dy\,dz + F_2\,dx\,dz + F_3\,dx\,dy$$

&

$$\vec{F} = \langle F_1, F_2, F_3 \rangle$$

So: $F_1 = x^3$

$F_2 = x^2y$

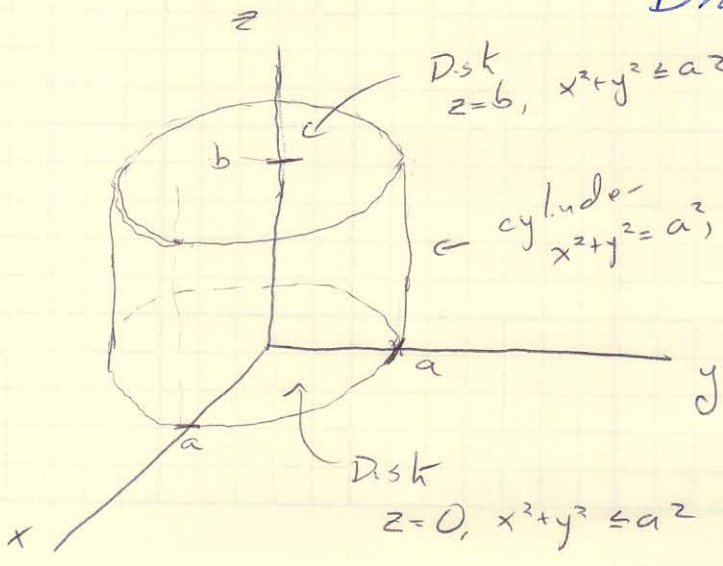
$F_3 = x^2z$

so $\vec{F} = \langle x^3, x^2y, x^2z \rangle$

Surface: S is the closed surface of $x^2+y^2=a^2$, $0 \le z \le b$ & the disks $z=0$, $z=b$, $x^2+y^2 \le a^2$

how many surfaces?

Draw it!!




At least 3 surfaces, but depending on coordinate system, might need 4

Bonus, why would you need 4 in cart coords → Think Orientation

So we can do 3 or 4 Surface integrals, or use The Divergence Theorem & do one triple integral,

Lets use Divergence Theorem

① Setup:

$\iiint \vec{\nabla} \cdot \vec{F} \, dV$ → Book uses polar, so I'll use cartesian

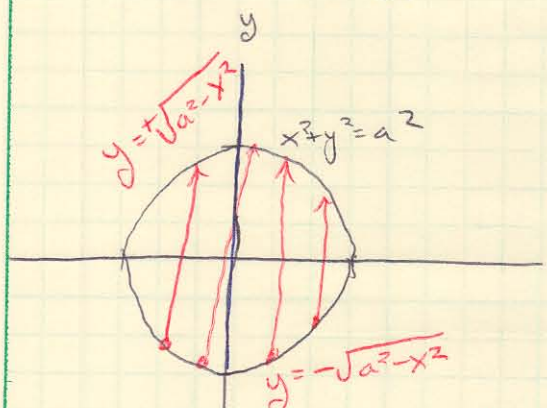
$\vec{F} = \langle x^3, x^2y, x^2z \rangle$

$\vec{\nabla} \cdot \vec{F} = 3x^2 + x^2 + x^2 = 5x^2$

so $\iiint \underbrace{5x^2}_{\vec{\nabla} \cdot \vec{F}} \underbrace{dy\,dx\,dz}_{dV \text{ in cartesian coordinates}}$

② Lets Find our limits

a. First lets look in the x-y plane, since y & x are related ($x^2+y^2=a^2$), but z is independent ($\underline{0 \le z \le b}$)




$$-a \le x \le a$$
$$-\sqrt{a^2-x^2} \le y \le +\sqrt{a^2-x^2}$$
$$0 \le z \le b$$

limits

time to refer back to MA205!! Type I vs Type II regions!

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③ Evaluate

$$\int_0^b \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 5x^2 \, dy\, dx\, dz$$

$$= \int_0^b \int_{-a}^{a} 5x^2 \, y \Big|_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dx\, dz$$

$$= \int_0^b \int_{-a}^{a} 5x^2 \left(2\sqrt{a^2-x^2}\right) dx\, dz$$

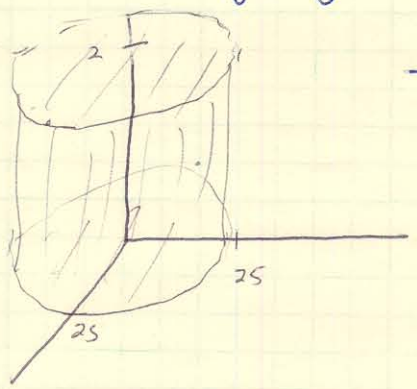
Neither u-sub or integration by parts simplifies this easily so use MMA

$$= 5/4\, a^4 b \pi$$

Another Ex: #20 in book:

Given $\vec{F} = \langle 3xy^2,\ yx^3 - y^3,\ 5zx^2 \rangle$

Surface:




→ 3 to 4 surface integrals

① Setup: $\vec{\nabla} \cdot \vec{F} = 3y^2 + x^2 - 3y^2 + 3x^2 = 4x^2$

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② Limits: $-25 \le x \le 25$

$-\sqrt{25-x^2} \le y \le +\sqrt{25-x^2}$

$0 \le z \le 2$

③ Solve: $a^4 b\pi = 5^4 \cdot 2\pi = 1250\pi$

How did I immediately jump to the solution?

Lets try one from last section using both surface integrals & the Divergence Theorem:

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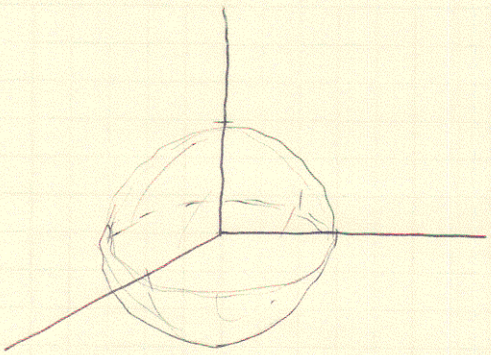
Given $\vec{F} = \langle 1, 1, 1 \rangle$

S: sphere: $x^2+y^2+z^2 = 1$

or

$z = \pm\sqrt{1-x^2-y^2}$



For surface integrals, we project into a plane:

since we wrote

$z = \pm\sqrt{1-x^2+y^2}$

This is actually two surfaces!!

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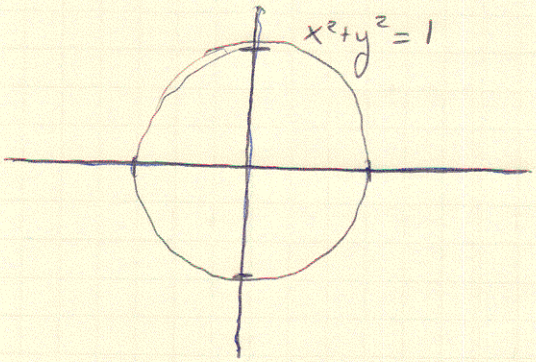
we will project into xy-plane:




Setting up our surface integral:

$$\iint_A \vec{F} \cdot \langle -f_x, -f_y, 1 \rangle \, dx\,dy$$

$$f_x = \frac{\partial}{\partial x}\left(\pm\sqrt{1-x^2-y^2}\right) = \frac{(\pm)\,-2x}{2(1-x^2-y^2)^{1/2}}$$

$$f_y = \frac{\partial}{\partial y}\left(\pm\sqrt{1-x^2-y^2}\right) = \frac{(\pm)\,-2y}{2(1-x^2-y^2)^{1/2}}$$

since we have two surfaces, we have two normals, so we treat seperately

Surface Integral 1: Top of sphere, Normal should be in $+z$ direction

$$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \langle 1, 1, 1 \rangle \cdot \left\langle \frac{2x}{\sqrt{1-x^2-y^2}}, \frac{y}{\sqrt{1-x^2-y^2}}, 1 \right\rangle dx\,dy$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{+x+y}{\sqrt{1-x^2-y^2}} + 1 \, dx\,dy$$

use u-sub & do by hand for first integral w/respect to x

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Surface Integral 2: Bottom of sphere, Normal should be pointing in $-z$ direction, so multiply $\vec{N}$ by $-1$

$$= \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{\mp x \; \mp y}{\sqrt{1-x^2-y^2}} \mp 1 \; dx \, dy$$

use u-sub, do by hand for integral w/ respect to $x$

using MMA $\rightarrow 0$

Now lets use D.U Theorem

$$\iiint_T \vec{\nabla} \cdot \vec{F} \, dV = \cancel{\iint} \vec{F} \cdot \hat{n} \, dS$$

$$\vec{\nabla} \cdot \vec{F} = \vec{\nabla} \cdot \langle 1, 1, 1 \rangle = 0$$

$$\iiint_T 0 \, dV = 0$$ For easier