11.1, 11.2 Fourier Series

$$F(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

sin nx & cos nx are orthogonal functions & can be used as building blocks for other functions.

How do we get the coefficients a_n & b_n?

For a period from $-\pi$ to π, the book says

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx$$

On Tuesday we found that the coefficients for $y = x$ were:

$$b_n = (-1)^n \frac{2}{n}$$

by minimizing sum of square error, Let's verify:

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{x \cdot \sin(nx)}\, dx$$

need to use integration by parts

$$\int u v' \, dx = uv - \int u' v \, dx$$

$u = x \qquad u' = 1$

$v' = \sin(nx) \qquad v = -\cos(nx)/n$

$$x(-\cos(nx))/n \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \cancel{-\cos(nx)/n}^{\,0} \, dx$$

$$\frac{1}{\pi} \cdot \left(\pi(-\cos(n\pi))/n - (-\pi(-\cos(n(-\pi))))/n \right)$$

$$= \frac{1}{\cancel{\pi}} \cdot \left(\cancel{\pi} \frac{(-\cos(n\pi))}{n} - \frac{\pi \cos(n\pi)}{n} \right)$$

$$= \frac{-2\cos(n\pi)}{n}$$

if n is even,

$\cos n\pi = 1$

if n is odd

$\cos n\pi = -1$

$$= \frac{-2 \cdot (-1)^n}{n}$$

$$b_n = \frac{2 \cdot (-1)^{n+1}}{n} \quad \checkmark$$

Finish the problem:

the Fourier series for $y = x$ from $-\pi$ to π is

$$f(x) = \sum_{n=1}^{\infty} \frac{2\cdot(-1)^{n+1}}{n} \cdot \sin(nx)$$

What if period $\neq 2\pi$?

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi}{L} x\right) + b_n \sin\left(\frac{n\pi}{L} x\right) \right)$$

look at the limits:
if $x \Rightarrow -L$ to L
then go for cos,
we go from

$\cos\left(\frac{n\pi(-L)}{L}\right)$ to $\cos\left(\frac{n\pi L}{L}\right)$
which is equivalent
to

$\cos(-n\pi)$ to $\cos(n\pi)$

This is equivalent
to what we had
for a period of
2π

For the coefficients:

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Example, Problem 4 pg 490

$f(x) = \frac{\pi x^3}{2}$ from -1 to 1

so $L = \cancel{2}\ 1$

now set up integrals:

$$a_0 = \frac{1}{2\cdot\cancel{1}}\int_{-1}^{1}\frac{\pi x^3}{2}\,dx$$ ← this is straight forward

$$= \frac{1}{2}\left(\frac{\pi x^4}{8}\right)\Big|_{-1}^{1}$$

$$= \frac{1}{2}\left(\frac{\pi}{8} - \frac{\pi}{8}\right) = 0$$

$$a_n = \frac{1}{\cancel{1}}\underbrace{\int_{-1}^{1} f(x)\cos\left(\frac{n\pi x}{L}\right)dx}$$

do I need to do this integral?

what type of function is $\frac{\pi x^3}{2}$? even or odd?

So all $a_n = 0$

$$b_n = \frac{1}{L}\int_{-L}^{L} \frac{\pi x^3}{2} \cdot \sin\left(\frac{n\pi x}{L}\right) dx$$

This requires 3 integration by parts → doable by hand, but lets let MMA do this one:

$$b_n = \frac{-n\pi(-6+n^2\pi^2)\cos(n\pi) + 3(-2+n^2\pi^2)\sin(n\pi)}{n^4\pi^3}$$

ouch! this is not a correct answer!! Simplify it!!

1st: $\sin(n\pi) = 0$ for all integer, n

so

$$b_n = \frac{-n\pi(-6+n^2\pi^2)\cos(n\pi)}{n^4\pi^3}$$

2nd: $\cos(n\pi) = ? = (-1)^n$

so

$$b_n = (-1)^n\left(\frac{-n\pi(-6+n^2\pi^2)}{n^4\pi^3}\right)$$

3rd: final bit of algebra

$$b_n = (-1)^n\left(\frac{-\cancel{n\pi}(-6+n^2\pi^2)}{n^{\cancel{4}3}\pi^{\cancel{3}2}}\right)$$

$$= (-1)^n\left(\frac{+6}{n^3\pi^2} - \frac{1}{n}\right)$$

So, the solution is:

$$f(x) = \sum_{n=1}^{\infty} \left((-1)^n \left(\frac{6}{n^3\pi^2} - \frac{1}{n} \right) \sin(n\pi x/2) \right)$$