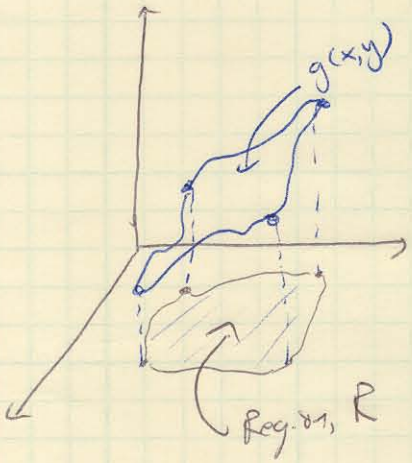
Green's Theorem

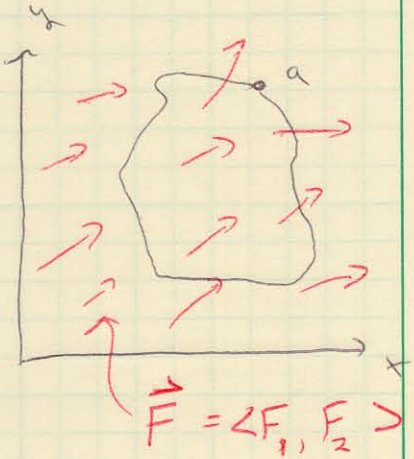
Green's Theorem gives us a way to transform between certain types of double integrals & closed path integrals

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\,dy = \oint_c F_1\,dx + F_2\,dy$$

where $\vec{F} = \langle F_1, F_2 \rangle$

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = g(x,y)$$





Bonus: What is R in this plot?

Green's Theorem:

$$\iint_R (\nabla \times \vec{F}) \cdot \vec{k}\,dx\,dy = \oint_c \vec{F} \cdot d\vec{r}$$

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Does it make sense?

Lets look @ a special case,

$\vec{F}$ is conservative:

$$\iint_R (\nabla \times \vec{F}) \cdot \vec{k}\, dx\, dy = \oint \vec{F} \cdot d\vec{r}$$

$(\nabla \times \vec{F}) = \vec{0}$ because $\vec{F}$ is conservative

$\oint \vec{F} \cdot d\vec{r} = 0$, this integral = 0, because $\vec{F}$ is conservative

so

$$\iint_R 0\, dx\, dy = 0 = \oint \vec{F} \cdot d\vec{r} = 0$$

why no constant? remember the constant is only for indefinite integrals, although R hasn't been defined, any definite integral makes it a constant

BONUS: why is it $(\nabla \times \vec{F}) \cdot \vec{k}$??

Lets work some examples:

#2, Sec 10.5

$\vec{F} = \langle y \sin x, 2x \cos y \rangle$, R = the square with vertices $(0,0), (\frac{1}{2}\pi, 0), (\frac{1}{2}\pi, \frac{1}{2}\pi), (0, \frac{1}{2}\pi)$

Find the work moving a particle around the square, counter-clockwise starting @ $(0,0)$

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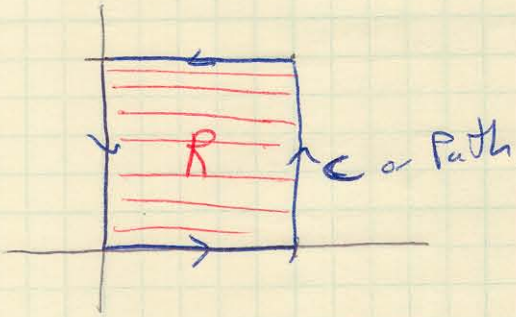
Step One: Plot The Region & Path



How many line integrals would this take?

By using Green's Theorem, we end up with a single double integral

Step two: Set up the double integral

Type 1 or Type 2 ??

$$\int_{\text{const}}^{\text{const}} \int_{g(y)}^{f(y)} dx\,dy \qquad \int_{\text{const}}^{\text{const}} \int_{g(x)}^{g(y)} dy\,dx$$

For this case either works well, so we'll use a type 1:

$$\int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} (\nabla \times \vec{F}) \cdot \vec{k}\; dx\,dy$$

curl is easy in 2-D !!

$\vec{\nabla} \times \vec{F}(x,y)$ in 2D

$= \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k}$

condition for exactness!! if = 0

$\vec{\nabla} \times \langle y \sin x, 2x \cos y \rangle$

$= (2\cos y - \sin x)\vec{k}$

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we now have:

$$\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} (2\cos y - \sin x)\underbrace{\vec{k}\cdot\vec{k}}\, dx\, dy$$

so that's why we use the $\cdot\vec{k}$ piece

$$= \int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} (2\cos y - \sin x)\, dx\, dy$$

Step three: evaluate the integral

$$\int_0^{\frac{1}{2}\pi}\left[\int_0^{\frac{1}{2}\pi} (2\cos y - \sin x)\, dx\right] dy$$

$$\int_0^{\frac{1}{2}\pi} (x 2\cos y + \cos x)\Big|_0^{\frac{1}{2}\pi}\, dy$$

$$= \int_0^{\frac{1}{2}\pi} \pi\cos y - 1\, dy$$

$$= \pi\sin y - y\Big|_0^{\frac{1}{2}\pi} = \pi - \tfrac{1}{2}\pi = \tfrac{1}{2}\pi$$

One more example (if time)

#12 sec 10.5

$\vec{F} = \langle x^2y^2, -x/y^2 \rangle$ R: $1 \le x^2+y^2 \le 4$, $x \ge 0$, $y \ge$ ~~0~~

$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = -\frac{1}{y^2} - 2x^2y$

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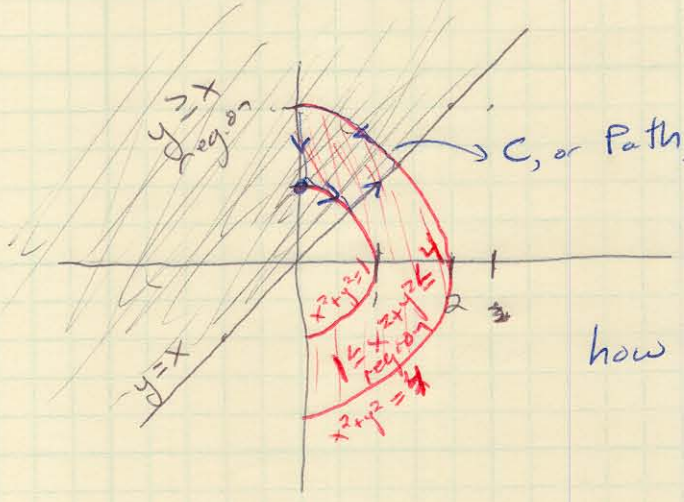
Step One: Draw the region!

$1 \le x^2+y^2 \le y$ → what does this give us? $x^2+y^2 = r^2$ so two circles

$x \ge 0$, $y \ge$ ~~0~~ → $y \ge x$ → $y = x$ is a line



so how many line integrals?

how many double integrals?

Lets do this in polar coordinates!

$x = r\cos\theta$
$y = r\sin\theta$
$dx\,dy = r\,dr\,d\theta$

} now we have a polar rectangular region:

$1 \le r \le 2$
$\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$

$$\int_{\pi/4}^{\pi/2}\int_{1}^{2}\left(\frac{-1}{r^2\sin^2\theta} - 2r^3\cos^2\theta\sin\theta\right) r\,dr\,d\theta$$

$$= \int_{\pi/4}^{\pi/2}\left[\int_{1}^{2} \frac{-1}{r\sin^2\theta} - 2r^4\cos^2\theta\sin\theta\,dr\right] d\theta$$

$$\int_{1}^{2} -\frac{1}{r\sin^2\theta} - 2r^4\cos^2\theta\sin\theta\,dr$$

$$= -\csc^2\theta \ln|r| - \frac{2r^5}{5}\cos^2\theta\sin\theta\Big|_1^2$$

$$= (-\ln 2)\csc^2\theta - \frac{2}{5}(32-1)\cos^2\theta\sin\theta$$

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$$\int_{\pi/4}^{\pi/2} -\ln(2)\csc^2\theta - \frac{62}{5}\cos^2\theta\sin\theta \, d\theta$$

$$\downarrow$$

$$-\ln 2\cot(\theta)\Big|_{\pi/4}^{\pi/2} - \frac{65}{15}\cos^3\theta\Big|_{\pi/4}^{\pi/2}$$

BIG BONUS

paragraph about Planimeter & how it relates to todays material!!

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