Heat Equation: Sec 12.5

What is the heat equation? Lets revisit sec 10.8 ex 2, where we modeled heat flow.

① From physical experiments, heat flows in the direction of decreasing temperature, and the rate of flow is proportional to the temperature gradient:

direction of decreasing tem

$$\underbrace{\vec{v}}_{\substack{\text{think of} \\ \text{as velocity} \\ \text{of heat flow}}} = \underbrace{(-)}_{}\underbrace{K}_{\substack{\text{proportionality} \\ \text{constant}}} \operatorname{grad}(\underbrace{U(x,y,z,t)}_{\substack{\text{temperature inside} \\ \text{body at any given} \\ \text{point \& time}}})$$

② let's imagine our object is bounded by a surface S, and find the amount of heat leaving our object:

$$\frac{\text{heat loss}}{\text{unit time}} = \underbrace{\oiint_S \underset{\text{heat flow}}{\vec{v}} \cdot \underset{\text{normal to surface}}{\vec{n}}\, dS}_{\text{sum up all the heat flowing out of surface}}$$

Since we have a closed surface, we can use the Divergence Theorem:

$$\oiint_S \vec{F} \cdot d\vec{S} = \iiint_V \vec{\nabla} \cdot \vec{F}\, dV$$

or

$$\oiint_S \underbrace{\vec{v} \cdot \vec{n}\, dS}_{\vec{F} \cdot d\vec{S}} = \iiint_V \underbrace{\vec{\nabla} \cdot \vec{v}}_{\vec{\nabla} \cdot \vec{F}}\, dV$$

Remember $\vec{v} = -K \operatorname{grad}(u)$

so $\vec{\nabla} \cdot \vec{v} = -K \cdot \underbrace{\vec{\nabla} \cdot \vec{\nabla}}_{\text{Laplacian operator}} u$

$$= -k \nabla^2 u$$

Finally

$$\frac{\text{heat loss}}{\text{unit time}} = \iiint_V -K \nabla^2 u \, dV$$

(3) Is there another way to find $\frac{\text{heat loss}}{\text{time}}$? what if we had a function for total amount of heat in object?

$$\underbrace{H}_{\text{total heat in object}} = \iiint_V \underbrace{\sigma}_{\text{specific heat}} \underbrace{\rho}_{\text{density}} \underbrace{u}_{\text{temperature function}} \, dV$$

but were not looking for Heat, we're looking for $\frac{\text{heat loss}}{\text{time}}$? what is this?

neg Rate of change of Heat/time

or

$$-\frac{dH}{dt}$$

$$-\frac{dH}{dt} = -\iiint_V \sigma \rho \frac{\partial u}{\partial t} \, dV$$

Notice that since we are integrating with respect to position, & σ & ρ are independent of time, The derivative w/respect to time carries through to u.

(4) At this point we have two equations for expessing $\frac{\text{heat loss}}{\text{time}}$ --- so what must be true?

They must be equal!

$$-\frac{\partial H}{\partial t} = -\frac{\partial H}{\partial t} \quad \text{... duh!}$$

$$-K\underbrace{\iiint_V}\nabla^2 u \underbrace{dV} = -\underbrace{\iiint_V}\sigma\rho \frac{\partial u}{\partial t}\underbrace{dV}$$

integrating over same volumes, so the integrands must be the same!

$$-K\nabla^2 u = -\sigma\rho\frac{\partial u}{\partial t}$$

Finally, let $c^2 = \frac{K}{\sigma\rho}$,

then

$$\boxed{\frac{\partial u}{\partial t} = c^2\nabla^2 u}$$ The heat equation

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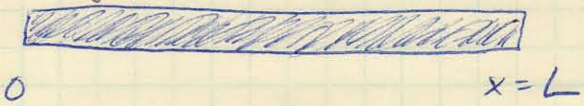
Not done yet -- Now we have to solve the heat equation -- This is a Partial Differential Equation because we have the derivative with respect to time & position -- the derivative w/respect to multiple variables -- ~~or~~ essential we have an equation relating the partial derivatives

As we have done in The past, lets start with a simple model:

A bar of length L, insulated everywhere except the ends



So u (temp) is only a function of $\underset{\text{position}}{x}$ & $\underset{\text{time}}{t}$

and we have the one dimensional heat equation:

$$\frac{\partial u}{\partial t} = c^2 \underbrace{\frac{\partial^2 u}{\partial x^2}}$$

remember: $\nabla^2 = \langle \frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial z^2} \rangle$, & since u is only a function of x, $\nabla^2 u = \frac{\partial^2 u}{\partial x^2}$

Let's see what we need to solve this equation:

$$\underbrace{\frac{\partial u}{\partial t}} = c^2 \underbrace{\frac{\partial^2 u}{\partial x^2}}$$

since we have the first time derivative, we will need an initial condition, $u(x,0) = ?$ = initial temp throughout bar

since we have the second x derivative, we will need two boundary conditions:

$u(0,t) = ?$ temp at $x=0$

$u(L,t) = ?$ temp at $x=L$

Example 1: Both ends of bar held @ 0°, ←

$u(x,0) = \underbrace{f(x)}$

we'll worry about what $f(x)$ is later, but

$f(0) = 0$, & $f(L) = 0$ ← from boundary conditions

Lets make an assumption about $u(x,t)$, if it doesn't work we can try something else:

Assume $u(x,t) = F(x)\,G(t)$

then we have

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left(F(x)\,G(t)\right) = F \cdot G'$$

and

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2}{\partial x^2}\left(F(x)\,G(t)\right) = F'' \cdot G$$

so

$$FG' = c^2 F'' G$$

rearranging so:

$$\underbrace{\frac{G'}{c^2 G}}_{\text{function of } t} = \underbrace{\frac{F''}{F}}_{\text{function of } x}$$

? when is the only time a function of time is equal to a function of position?

IF they both equal a constant!

so

$$\frac{G'}{c^2 G} = \frac{F''}{F} = K$$

we can now split this up into two Ordinary Differential Equations

$$\frac{G'}{c^2 G} = K \qquad \frac{F''}{F} = K$$

but $K < 0$, otherwise $\underbrace{F = G = u = 0}_{\text{uninteresting solution}}$

so let $K = -p^2$ → For any p, $K < 0$

$$\frac{G'}{c^2 G} = -p^2$$

solving:

$$\underbrace{G' + c^2 p^2 G = 0}_{\text{pause}}$$

solve first

Now using $p = \frac{n\pi}{L}$

$$\underbrace{G' + \frac{c^2 n^2 \pi^2}{L^2} G = 0}_{\text{seperable diff eq}}$$

let $\lambda_n = \frac{cn\pi}{L}$

$$G(t) = Ce^{-\lambda_n^2 t}$$

Note: we have different solutions for different values of n, how do we know what value of n to use??

$$\frac{F''}{F} = -p^2$$

solving:

$F'' + p^2 F = 0$ } 2nd Order Cnst coefficient Diff EQ solve w/characteristic equation

$$F = A\cos px + B\sin px$$

Recall our boundary conditions:

$u(0,t) = 0$, $u(L,t) = 0$

so

$u(0,t) = F(0)G(t) = 0$

$u(L,t) = F(L)G(t) = 0$

For $G \neq 0$

$F(0) = 0$, $F(L) = 0$

$F(0) = \underline{A = 0}$

$F(L) = B\sin(p\cancel{x}L) = 0$

if $B = 0$, $\underbrace{\text{the } F = 0, \therefore u = 0}_{\text{not interested}}$

so

$B \neq 0$

$B\sin(p\cancel{x}L) = 0$ when?

~~when does~~ ~~sin(a) = sin = 0~~

when does $\sin(a) = 0$

$a = n\pi$

so $\sin(p\cancel{x}L) = 0$ when

$px = n\pi$, or $p = \frac{n\pi}{L}$

Let's put it all togeter:

$G = C_n e^{-\lambda_n^2 t}$ where $\lambda_n = \frac{cn\pi}{L}$

$F = B \sin\left(\frac{n\pi x}{L}\right)$

$$u_n(x,t) = \underbrace{B \cdot C_n}_{\text{simplify into one constant, say } B_n} \sin\left(\frac{n\pi x}{L}\right) e^{-\lambda_n^2 t}$$

$$u_n(x,t) = B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\lambda_n^2 t}$$

Now how do we determine n?

What's the one piece of info we haven't used? $\Rightarrow$ The initial condition

Let: $u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$

& we know $u(x,0) = f(x)$,

so

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \cdot \overset{1}{\cancel{e^{-\lambda_n^2 \cdot 0}}}$$

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

we now have a Fourier series problem, so for a given $f(x)$, we find the B_n coefficients to complete the problem:

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\lambda_n^2 t}$$

$\cancel{\lambda_n} = \frac{\cancel{c}n\pi}{L}$ = eigen values of equation

& $\sin\left(\frac{n\pi x}{L}\right)$ = eigen function