Applications of Eigenvalues

Stretching

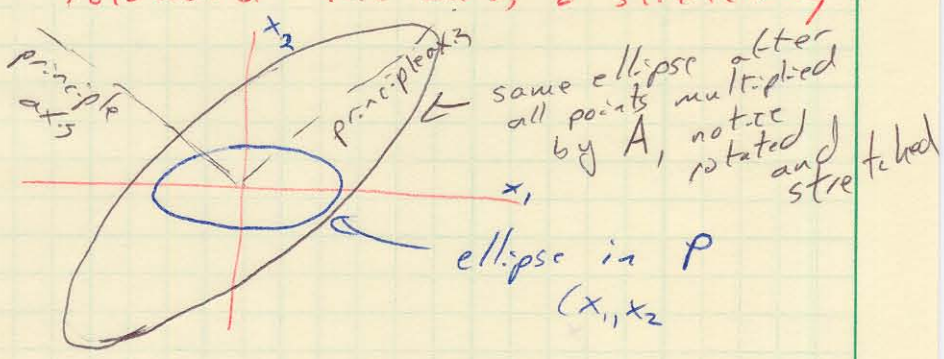
Given a mapping of:

$$\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = A\vec{x} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

in otherwords we are mapping or changing any point in $P_i(x_1, x_2)$, to a new point in $Q_i(y_1, y_2)$

Find the principle directions, the directions of the position vector $\vec{x}$ of P, for which the position vector $\vec{y}$ of Q is the same or exactly opposite

↳ think of it as finding the rotation of the axis, & stretching



Lets think about the setup of this problem so the goal is to find the direction of position vector $\vec{x}$ for which the position vector $\vec{y}$ of Q is same or opposite:

$$\vec{y} = A\vec{x}$$

but want $\underbrace{\vec{x} = \vec{y}}_{\text{same}}$ or $\underbrace{-\vec{y}}_{\text{opposite}}$

So we have

$$A\vec{x} = \lambda\vec{x}$$

looks like an eigenvalue problem

we won't assume $\lambda = 1$ or -1 we'll discuss later, but we are primarily concerned only with dir. not magn. tude

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 2\lambda - 3 = 0$$

$$\lambda = 3, -1$$

Find the corresponding vectors:

$\lambda = 3$

$$\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$$

Eigenvector $= \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = -1$

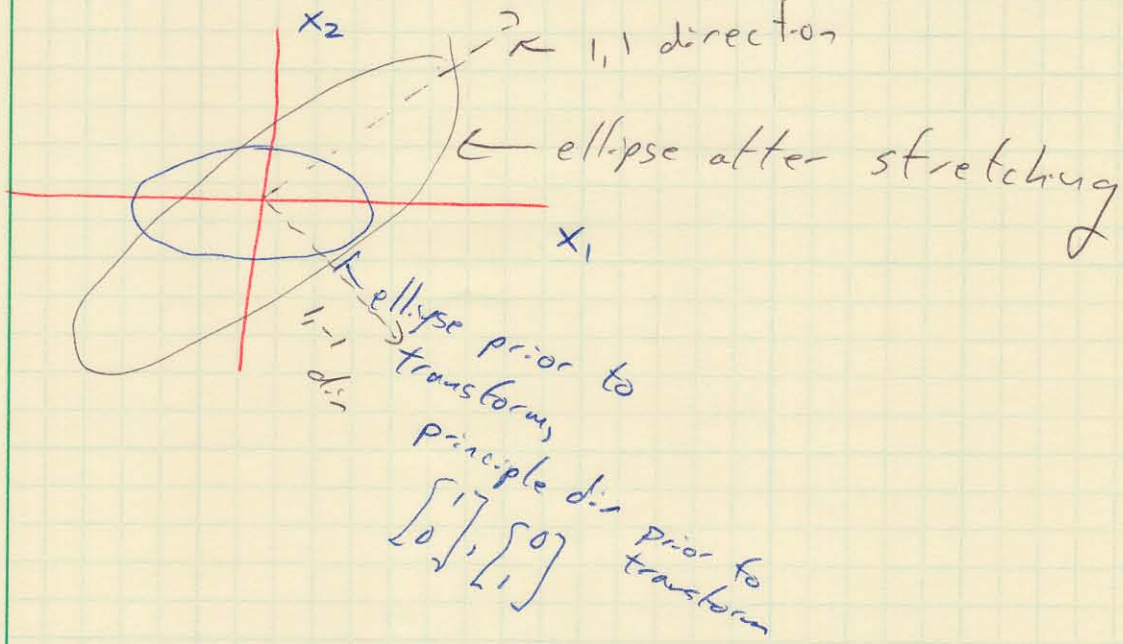
$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$$

Eigenvector $= \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

So we have:

$$\lambda = 3, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \lambda = -1, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

principle directions of our rotation, note the vectors are orthogonal



What happens if our eigen vectors are non-orthogonal?

Bonus: Use the following code in MMA and rotate the ellipse $x[t] = \{3\cos[t], 2\sin t]\}$ 30°, 45°, 90° & ~~225~~ 240°

theta = _____

code

$$A = \begin{bmatrix} \text{Cos}[\text{theta}] & -\text{Sin}[\text{theta}] \\ \text{Sin}[\text{theta}] & \text{Cos}[\text{theta}] \end{bmatrix}$$

send me your work

```
EigenSystem[A] // N
x[t_] = {3*Cos[t], 2*Sin[t]};
y[t_] = A . x[t]
ParametricPlot[{{x[t][[1]], x[t][[2]]}, {y[t][[1]], y[t][[2]]}}, {t, 0, 2*Pi}]
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