Lesson 10 - Short Impulses

what is an impulse??

From wikipedia, an impulse is defined as the integral of force with respect to time, or a simpler form is

$$\boxed{F\Delta t = I} = m\Delta v$$

→ change in momentum

Force

short period of time the force is acting on an object

how do we model this??

Lets define:

$$F_k(t-a) = \begin{cases} 1/k & a \le t \le a+k \\ 0 & \text{otherwise} \end{cases}$$

huh?

This essentially models a force of $1/k$ acting from $t=a$ to $t=a+k$

The integral of this function is the impulse

$$I_k = \int_0^\infty F_k(t-a)\,dt = \int_a^{a+k} \frac{1}{k}\,dt = 1$$

In engineering we often look for the simplest model that will accurately describe our scenario, we can model impulses with the $\boxed{\text{Dirac Delta}}$ function

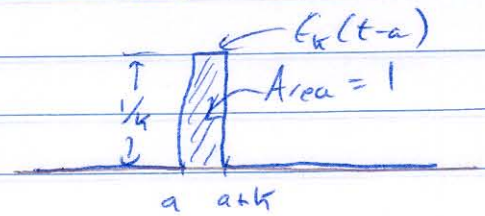
$$\delta(t-a) = \lim_{k \to 0} f_k(t-a) \quad \rightarrow \text{also called unit impulse}$$

$$\delta(t-a) = \begin{cases} \infty & \text{if } t=a \\ 0 & \text{otherwise} \end{cases} \quad \& \quad \int_0^\infty \delta(t-a)\,dt = 1$$

If we plan to use this function w/ Differential equations, we need to know its Laplace transform:

Lets look first at f_k:

$$f_k(t-a) = \frac{1}{k}\left[u(t-a) - u(t-(a+k))\right]$$



$$\mathcal{L}\{f_k(t-a)\} = \frac{1}{k}\left[\frac{e^{-as}}{s} - \frac{e^{-(a+k)s}}{s}\right]$$

$$= \frac{1}{ks}\left[e^{-as}\left(1 - e^{-ks}\right)\right]$$

Now we want to get to the Dirac Delta transform - How

To get the Dirac Delta, we take the limit as $k \to 0$

$$\mathcal{L}\{\delta(t-a)\} \quad \underbrace{\lim_{k\to 0} e^{-as} \frac{(1-e^{-ks})}{ks}} = \frac{0}{0}??$$

what now?

L'Hopital's rule:
take the limit of the derivatives of the numerator & denominator.

$$\Downarrow$$

$$e^{-as} \lim_{k\to 0} \frac{(1-e^{-ks})'}{(ks)'}$$

$$\Downarrow$$

$$e^{-as} \lim_{k\to 0} \frac{se^{-ks}}{s} = e^{-as}(1)$$

thus:

$$\mathcal{L}\{\delta(t-a)\} = e^{-as}$$

Ok, so lets use this:

if I drop my model car from 4ft, what impulse will it experience?

$\underline{I = m\Delta v}$

need to know the change in ~~mass~~ velocity, with end velocity $= 0$, so need to know how fast it hit the ground, an the mass.

Mass = ? model weighs $\simeq$ 12 lbs
so $m \cong 5.5$ kg

Velocity = ? Let's set up an ODE
(equations of motion)
neglecting air resistance

$$mx'' = m \cdot g \qquad x(0) = 0$$
$$x'' = 9.8 \qquad x'(0) = 0$$
$$x' = 9.8t$$
$$\boxed{x(t) = 4.9t^2}$$

→ this gives us position, so find t when x = 4

$$4 = 4.9t^2$$
$$t = .9 \text{ secs}$$

→ now find ~~t~~ v at t = .9 sec

$$v = x'(t) = 9.8t$$
$$= 8.82 \text{ m/s} \quad (\approx 20 \text{ mph})$$

Now we can find I:

$$I = m\Delta v = 5.5 \cdot 8.82 = 48.5 \text{ Ns}$$

All this is set up:

Now lets see how the suspension handles this drop?

Model:

$$my'' + cy' + ky = 48.5\,\delta(t-a)$$

$m = 5.5$ $\qquad a = \cancel{0}\ 1 \rightarrow$ although we could use zero, 1 is slightly more interesting

$k = .836$ N/m

$c = ?$ start with $c = \cancel{2}\ 5$

$$\mathcal{L}\{5.5y'' + 2y' + .8y = 48.5\,\delta(t-1)\} \qquad y(0)=0,\ y'(0)=0$$

$$5.5s^2 Y(s) + 2s\,Y(s) + .8\,Y(s) = 48.5e^{-s}$$

$$Y(s) = \frac{48.5e^{-s}}{5.5s^2 + \underset{5}{\cancel{2}}s + .8}$$

$$\mathcal{L}^{-1}\left\{\frac{48.5e^{-s}}{5.5s^2 + \underset{5}{\cancel{2}}s + .8}\right\} \Rightarrow \mathcal{L}^{-1}\left\{\overbrace{\underbrace{48.5e^{-s}}_{\text{this causes t-shifting}}}^{e^{-as}} \cdot \overbrace{\frac{1}{5.5s^2 + \underset{5}{\cancel{2}}s + .8}}^{F(s)}\right\}$$

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = u(t-a)\,\overbrace{\mathcal{L}^{-1}\{F(s)\}}^{\text{time shifted}}$$

$$= u(t-a)f(t-a)$$

so let's find $\mathcal{L}^{-1}\{F(s)\}$

$$F(s) = \frac{1}{(s+.21)(s-.7)}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+.21)(s+.7)}\right\} = \mathcal{L}^{-1}\left\{\underbrace{\frac{A}{(s+.21)} + \frac{B}{(s+.7)}}_{\text{partial Fractions!}}\right\}$$

$$\frac{1}{(s+.21)(s+.7)} = \frac{A}{s+.21} + \frac{B}{s+.7}$$

Mult by $(s+.21)$ & let $s = -.21$

$$\frac{1}{.49} = A \cong 2$$

Mult by $(s+.7)$ & let $s = -.7$

$$\frac{1}{-.49} = B \cong -2$$

$$\mathcal{L}^{-1}\left\{\frac{2}{s+.21} + \frac{-2}{s+.21}\right\} = 2e^{-.21t} - 2e^{-.7t} = F(t)$$

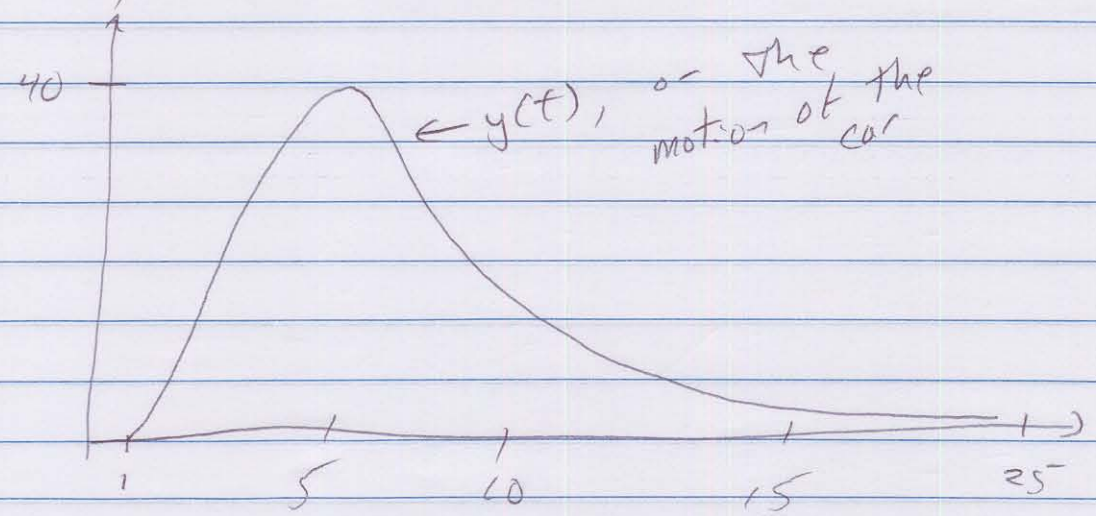
This is Not $F(t-a)$

Use the above w/ $\mathcal{L}^{-1}\{e^{-as}F(s)\} = u(t-a)F(t-a)$ and we get:

$$y(t) = 48u(t-1)\left(2e^{-.21(t-1)} - 2e^{-.7(t-1)}\right)$$

There are a lot of simplifications here, but it gives a good example of one way to model a system

Plotting the solution we get:



Bonus: Does this solution make sense? explain (think about the graph & associated units)

I'll bring in the car & we'll actually drop it & watch.