Matrices, Systems, Determinants

Lets start off by defining a few terms

Linear independence: ~~Our vectors, equations, rows~~ ~~all the same~~

There is no scalar combination of the vectors that sums to $\vec{0}$
equations, rows

Rank: The rank of a matrix is the maximum number of linearly independent row vectors

← How do I find it: Gaussian Elimination

Overdetermined System: More equations than unknowns

Underdetermined System: More unknowns than equations

Solutions to Linear Systems:

Given the following system of equations:

$$3x + 2y + z = 4$$
$$6x + y + 2z = 6$$

Find a solution:

① Setup in Augmented Matrix Form

$$\left[\begin{array}{ccc|c} \multicolumn{3}{c}{\text{coefficients Matrix}} & \text{solution vector} \\ 3 & 2 & 1 & 4 \\ 6 & 1 & 2 & 6 \end{array}\right]$$

(2) Perform Gauss elimination until coefficient matrix is in upper-triangular form

$\xrightarrow{}$ pivots

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 4 \\ 6 & 1 & 2 & 6 \end{array}\right]$$

upper Δ form:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & \vdots \\ & & \ddots & \vdots \\ 0 & & & a_{nn} \end{bmatrix}$$

all zeros below diagonal

a. 1st Operation:

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 4 \\ 6 & 1 & 2 & 6 \end{array}\right] 2R_1 - R_2 = R_2$$

$$= \left[\begin{array}{ccc|c} 3 & 2 & 1 & 4 \\ 0 & 3 & 0 & 2 \end{array}\right]$$

Do we go any further? No

b. Solve for y:

$$y = 2/3$$

Solve for x

$$x = 4 - 4/3 - \underline{\underline{z}}\,?$$

Can we find a z? No, system is under determined

(3) Find $\underline{\underline{a}}$ solution:

let
$$\boxed{\begin{aligned} z &= 0 \\ y &= 2/3 \\ x &= 4 - 4/3 - 0 = 8/3 \end{aligned}}$$

Determinants → needed for Eigen value/vector problems!

Determinants were originally ~~for~~ introduced for solving linear systems (from book)
^ Bad idea, computationally expensive

but they have important engineering applications in Eigen values problems!

Let $D = \det A = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$ → A must be square!!

straight brackets represent det

how do we find D?

Lets start w/a 2×2:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$D = \det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Once we can find the det of A 2×2, we can find it for any size matrix

Ex: Find $\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 2 \cdot 2 - 1 \cdot 1 = 3$

Now lets look at a 3×3

Ex: Find $\begin{vmatrix} 3 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{vmatrix}$

How?
③ we will do this through cofactor expansion

① choose any row or column.

$$\begin{vmatrix} 3 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{vmatrix}$$

I chose row 1

② Next select the first element in your chosen row or column

$$\begin{vmatrix} 3 & \cancel{1} & \cancel{1} \\ \cancel{0} & 1 & 1 \\ \cancel{0} & 2 & 2 \end{vmatrix}$$

and now pull out the submatrix that does not correspond to the row & column the chosen element is in.

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

(3) Multiply the determinant of that submatrix by ~~you~~ the chosen element:

$$3\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 3(1\cdot 2 - 2\cdot 1) = 0$$

(4) Assign a sign (+ or −) based on the location of the element:

Method 1:

for a 3×3

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

Method 2:

$$(-1)^{j+k}$$

where j is the row of the element & k is the column

$$(-1)^{1+1} = 1 = +$$

(5) repeat process for every element in chosen row:

$$\begin{vmatrix} \cancel{3} & 1 & \cancel{1} \\ 0 & \cancel{1} & 1 \\ 0 & \cancel{2} & 2 \end{vmatrix}$$

$$1\cdot\begin{vmatrix} 0 & 1 \\ 0 & 2 \end{vmatrix} = 0$$

sign $(-1)^{1+2} = -$

$$\begin{vmatrix} \cancel{3} & \cancel{1} & 1 \\ 0 & 1 & \cancel{1} \\ 0 & 2 & \cancel{2} \end{vmatrix}$$

$$1\cdot\begin{vmatrix} 0 & 1 \\ 0 & 2 \end{vmatrix} = 0$$

sign $= (-1)^{1+3} = +$

(6) Sum up the pieces:

$$+0 - 0 + 0 = 0$$

Second Example:

Find:
$$\begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 3 & 1 & 1 \end{vmatrix}$$

? does it matter which row or column I choose? NO

(1) Whats a good choice for our row or column?

$$\begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 3 & 1 & 1 \end{vmatrix}$$

two zeros, so all I need to look @ is the middle piece:

$$(-1)^{2+2} \cdot 1 \cdot \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} = 1 \cdot (2-3) = \underline{\underline{-1}}$$

Easy when there are a lot of zeros.

one last Example:

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

(1) ~~wh~~ Choose a row & expand

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$\leftarrow a_{4,4}$

$$(-1)^{4+4} \cdot 1 \cdot \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix}$$

This is a 3×3, so we have to expand again on this smaller matrix

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix}$$

$$(-1)^{3+3} \cdot 1 \cdot \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}$$

$$= 1 \cdot (1 \cdot 1 - 0)$$

$$= 1$$

$(-1)^{4+4} \cdot 1 \cdot 1 = 1$

or if we follow the math & realize all signs are positive, we get

$1 \cdot 1 \cdot 1 \cdot 1 = 1$

Now For Bonus Find:

$$\begin{vmatrix} 5 & 4 & 3 & 2 & 1 \\ 0 & 4 & 3 & 2 & 1 \\ 0 & 0 & 3 & 2 & 1 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

hint, look at the above example

show me the work in your email

Cramers Rule:

A terrible way to solve a linear system of equations: (numerically unstable!)

Given:

$$\underbrace{\begin{matrix} a_{11} x_1 + a_{12} x_2 + \cdots a_{1n} x_n = b_1 \\ \vdots \\ a_{n1} x_1 + a_{n2} x_2 + \cdots a_{nn} x_n = b_n \end{matrix}}_{\text{system of equations}}$$

$$A\vec{x} = \vec{b}$$

$$x_1 = \frac{D_1}{D}, \quad x_2 = \frac{D_2}{D}, \quad x_3 = \frac{D_3}{D}, \quad \dots \quad x_n = \frac{D_n}{D}$$

where $D = \det(A)$

& D_1 is the

$$\det\begin{pmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

& D_2 is the

$$\det\begin{pmatrix} a_{11} & b_1 & a_{13} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & b_n & a_{n3} & \cdots & a_{nn} \end{pmatrix}$$

etc.

Example: Solve

$$3x + 4y = 3$$
$$2x + 5y = 1$$

$D = \begin{vmatrix} 3 & 4 \\ 2 & 5 \end{vmatrix} = 3 \cdot 5 - 2 \cdot 4 = 7$

$D_1 = \begin{vmatrix} 3 & 4 \\ 1 & 5 \end{vmatrix} = 3 \cdot 5 - 4 \cdot 1 = 11$

$D_2 = \begin{vmatrix} 3 & 3 \\ 2 & 1 \end{vmatrix} = 3 \cdot 1 - 2 \cdot 3 = -3$

$x_1 = \frac{11}{7}$

$x_2 = \frac{-3}{7}$

Inverse of A matrix

The inverse can be used to solve a linear system of equations as follows:

Given: $A\vec{x} = \vec{b}$

then: $\vec{x} = A^{-1}\vec{b}$

This is a computationally expensive & numerically unstable technique.

What is A^{-1}?

A^{-1} is the matrix that when multiplied by A, gives the Identy matrix:

$$A^{-1}A = I = AA^{-1}$$

$$B^{-1}B = I = BB^{-1}$$

An inverse exists if the rank of A is n, thus, if & only if the $\det(A) \neq 0$

~~Find~~

Finding an Inverse:

Given $A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$, find A^{-1}

① Create an agumented Matrix:

$$[\underline{\underline{A}} \mid \underline{\underline{I}}]$$

$$= \left[\begin{matrix} 2 & 1 \\ 1 & 3 \end{matrix} \middle| \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right]$$

② Perform row operations until A is an Identity matrix:

$$\left[\begin{matrix} 2 & 1 \\ 1 & 3 \end{matrix} \middle| \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right] \quad R_1 - 2R_2 = R_2$$

$$\left[\begin{matrix} 2 & 1 \\ 0 & -5 \end{matrix} \middle| \begin{matrix} 1 & 0 \\ 1 & -2 \end{matrix}\right] \quad R_2 = R_2/-5$$

$$\left[\begin{matrix} 2 & 1 \\ 0 & 1 \end{matrix} \middle| \begin{matrix} 1 & 0 \\ -1/5 & 2/5 \end{matrix}\right] \quad R_1 = R_1 - R_2$$

$$\left[\begin{matrix} 2 & 0 \\ 0 & 1 \end{matrix} \middle| \begin{matrix} 6/5 & -2/5 \\ -1/5 & 2/5 \end{matrix}\right] \quad R_1 = R_1/2$$

$$\left[\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \middle| \begin{matrix} 6/10 & -2/10 \\ -2/10 & 4/10 \end{matrix}\right]$$

③ The inverse is now where the Id Matrix was:

$$A^{-1} = \begin{bmatrix} 6/10 & -2/10 \\ -2/10 & 4/10 \end{bmatrix}$$