Eigenvalues & Vectors

Sec 8.1

→ First we will develop the techniques for finding Eigenvalues & vectors, then we will discuss applications

Eigen values arise from problems like this:

$$A\vec{x} = \lambda\vec{x}$$

where A is an $n \times n$ matrix, $\vec{x}$ is an unknown vector, and λ is an unknown scalar that satisfys the equation.

λ is called the eigenvalue
$\vec{x}$ is called the eigen vector

Note: we are not concerned with the sol $\vec{x} = \vec{0}$

Ex: Lets look @ a 2×2 case

Given $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ find the eigen values & eigen vectors of A.

Lets look @ the eq:

$$A\vec{x} = \lambda\vec{x}$$

Lets get all the $\vec{x}$ on the same side of the eq:

$$A\vec{x} - \lambda\vec{x} = 0$$

Can we factor out $\vec{x}$?
yes but we must replace $\vec{x}$ in the $\lambda\vec{x}$ piece with I

Bonus: Why? be detailed

$$\underbrace{(A - \lambda I)\vec{x} = 0}$$

at this point we see this is a homogeneous linear system
By Cramers Theorem (sec 7.7) it has a nontrivial sol $\vec{x} \neq \vec{0}$ if & only if its coefficient determinant is 0

So we set:

$$|\underbrace{A - \lambda I}| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$(1-\lambda)(1-\lambda) - 4 = 0$ ← solve this eq

$\lambda^2 - 2\lambda - 3 = 0$ → $D(\lambda)$ → called the characteristic polynomial

$(\lambda+1)(\lambda-3) = 0$

$\lambda = -1, 3$ → These are the eigen values of A.

Next find the corresponding eigen vectors to each eigen value:

<u>start with $\lambda = -1$</u>

$$(A - \lambda I)\vec{x} = 0$$

$$\begin{bmatrix} 1--1 & 2 \\ 2 & 1--1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$$

$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$$

Using Gauss Elimination:

$R_2 = R_1 - R_2$

$$\left.\begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$$

underdetermined this is required!

$x_2 = -x_1$ → so an eigenvector is:

$x_1 = 1, \quad x_2 = -1$

$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ for $\lambda = -1$

Now for $\lambda = 3$:

$$\begin{bmatrix} 1-3 & 2 \\ 2 & 1-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$$

$$\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$$

$R_2 = R_1 + R_2$

$$\left.\begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}\right\}$$

once again underdetermined

$x_1 = x_2$

eigenvector is:

$x_1 = 1, \quad x_2 = 1$

or $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = -1, \quad \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda = 3, \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

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More Eigenstuff

Thus 142 out of book

a. The eigen values are the roots of the characteristic equation of A, hence, an $n \times n$ matrix has @ least one eigenvalue & at most n numerically different eigenvalues

b. if $\vec{w}$ & $\vec{x}$ are eigenvectors of A corresponding to some eigenvalue λ then:

$$\vec{w} + \vec{x}$$
$$\vec{w} - \vec{x}$$
$$k\vec{x}$$
$$l\vec{w}$$
$$(\vec{w} + k\vec{x}$$

are all eigenvectors of A corresponding to λ

Hence the ~~eigenvalues~~ eigen vectors corresponding to the same eigen value λ of A along with $\emptyset$ form a vector space, called the eigen space of A to that λ.

Multiple Eigen values

ex: #17

$$A = \begin{bmatrix} 4 & -2 & 3 \\ -2 & 1 & 6 \\ 1 & 2 & 2 \end{bmatrix}$$

Find eigen values & vectors of A

Give $\lambda_1 = -3$

$$|(A - \lambda I)| = 0$$

$$\begin{vmatrix} 4-\lambda & -2 & 3 \\ -2 & 1-\lambda & 6 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 4-\lambda \begin{vmatrix} 1-\lambda & 6 \\ 2 & 2-\lambda \end{vmatrix} + 2 \begin{vmatrix} -2 & 6 \\ 1 & 2-\lambda \end{vmatrix} + 3 \begin{vmatrix} -2 & 1-\lambda \\ 1 & 2 \end{vmatrix}$$

$$= 4-\lambda((1-\lambda)(2-\lambda) - 12) + 2(-2(2-\lambda) - 6) + 3(-4 - 1(1-\lambda))$$

$$=$$

$= (4-\lambda)(\lambda^3 - 3\lambda + 2 - 12) + 2(-4 + 2\lambda - 6) + 3(-4 - 1 + \lambda)$

$= (4-\lambda)(\lambda^2 - 3\lambda - 10) + (4\lambda - 20) + (3\lambda - 15)$

$= 4\lambda^2 - 12\lambda - 40 - \lambda^3 + 3\lambda^2 + 10\lambda + 4\lambda - 20 + 3\lambda - 15$

$-\lambda^3 + 7\lambda^2 + 5\lambda - 75 = 0$

since we know $\lambda = -3$ is one root, we can factor out a $(\lambda + 3)$

$$
\begin{array}{r}
-\lambda^2 + 10\lambda - 25 \\
(\lambda+3)\overline{)\,-\lambda^3 + 7\lambda^2 + 5\lambda - 75} \\
\underline{-\lambda^3 - 3\lambda^2} \\
10\lambda^2 + 5\lambda - 75 \\
\underline{10\lambda^2 + 30\lambda} \\
-25\lambda - 75 \\
-25\lambda - 75
\end{array}
$$

$(\lambda+3)(-\lambda^2 + 10\lambda - 25) = 0$

$-(\lambda+3)(\lambda^2 - 10\lambda + 25) = 0$

$-(\lambda+3)(\lambda - 5)^2 = 0$

$\lambda = -3, \underbrace{5, 5}$ → repeated values of λ, how will this affect my eigen vectors?

Lets look at $\lambda = 5$, & find the eigenvectors

$(A - \lambda I)\vec{x} = \vec{0}$

$$\begin{bmatrix} 4-5 & -2 & 3 \\ -2 & 1-5 & 6 \\ 1 & 2 & 2-5 \end{bmatrix} \vec{x} = \vec{0}$$

whats the rank of this matrix rank refers to ly # of linearly ind rows

$$\begin{bmatrix} -1 & -2 & 3 \\ -2 & -4 & 6 \\ 1 & 2 & -3 \end{bmatrix} \vec{x} = \vec{0}$$

$R_2 = R_2 - 2R_1$

$R_3 = R_3 + R_1$

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$$\begin{bmatrix} -1 & -2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{\emptyset}$$

so

$$x_1 = \underbrace{3x_3 - 2x_2}$$

two free variables so we can get two linearly independent vectors:

let $x_3 = 1$, $x_2 = 1$,

then $x_1 = 1$

so 1st vector is:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

what if we choose $x_2 = -1$ & $x_3 = -1$,

then we get $x_1 = -1$,

Bonus is $\begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$ a seperate linearly independent eigenvector?

if not, find a second linearly independent eigenvector, for $\lambda = 5$

Multiplicity of eigenvalues:

Algebraic Multiplicity of λ: M_λ = algabraic Mult., is the # of times λ is repeated. In the above example, $M_\lambda = 1$ for -3, & $M_\lambda = 2$ for $\lambda = 5$.

Geometric Multiplicity: m_λ is the geometric multiplicity and corresponds to the # of linearly independent eigen vectors for a given λ.

$M_{-3} = 1, \quad m_{-3} = 1$

$M_5 = 2, \quad m_5 = 2$

When $M_\lambda = m_\lambda$ there are no <u>defects</u>.

in general $\quad m_\lambda \leq M_\lambda$

The # of defects is $\quad \Delta_\lambda = M_\lambda - m_\lambda$

ex: $\begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix}$

Work it out on your own