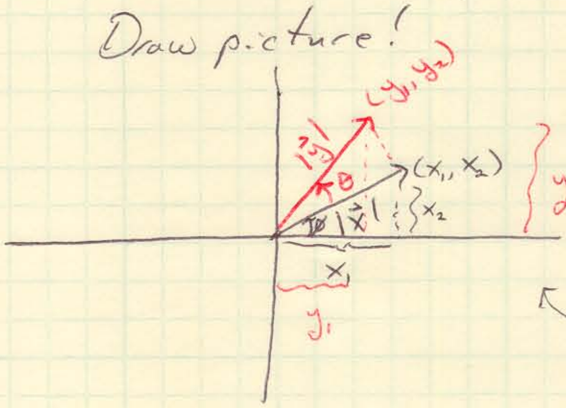
Applications of Eigenvalues:
Systems of ODEs

1^{st}: Additional notes from prev lesson:

Rotation Matrix:

Find a matrix **A**, such that it rotates a vector by θ.

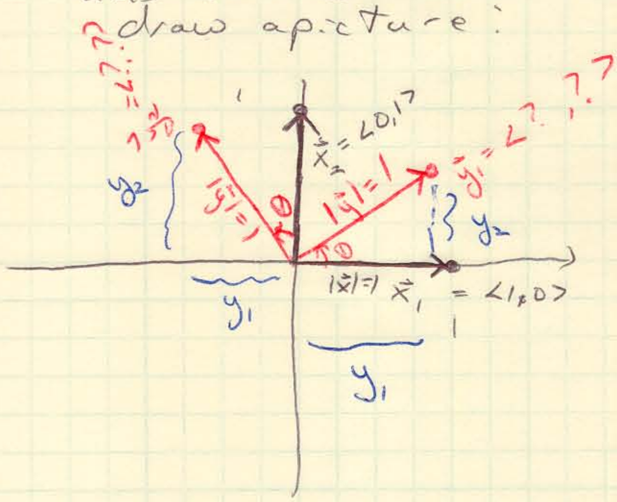
Draw picture!



Lets set up the relation between $\vec{x}$ & $\vec{y}$ by individual components:

We can do it from this set up, but is there an easier way?

Lets look @ the $\vec{x} = \langle 1, 0 \rangle$, & $\langle 0, 1 \rangle$, now draw a picture:



For $\vec{y}_1$:

$y_1 = \cos\theta \cdot 1$

$y_2 = \sin\theta \cdot 1$

For $\vec{y}_2$:

$y_1 = -\sin\theta \cdot 1$

$y_2 = \cos\theta \cdot 1$

remember soh cah toa

Now lets use that info to setup our equations:

we wanted to Find **A** such that

$A\vec{x} = \vec{y}$, where $\vec{y}$ is the rotation in θ degrees of $\vec{x}$.

By using two known $\vec{x}$s, we came up with:

$$\mathbf{A}\overset{\vec{x}}{\begin{bmatrix}1\\0\end{bmatrix}} = \overset{\vec{y}}{\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}}, \quad \mathbf{A}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$$

but **A** is $\begin{bmatrix}a & b\\ c & d\end{bmatrix}$, now we have

four unknowns & four equations

$$\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}, \quad \begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$$

$$1\cdot a + 0\cdot b = \cos\theta \qquad 0\cdot a + 1\cdot b = -\sin\theta$$
$$1\cdot c + 0\cdot d = \sin\theta \qquad 0\cdot c + 1\cdot d = \cos\theta$$

so: $a = \cos\theta$ $\quad b = -\sin\theta$
$c = \sin\theta$ $\quad d = \cos\theta$

giving us:

$$\mathbf{A} = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$$

Ex: Find principle direction of A!

two ways → brute Force & Reason:

Brute Force

Principle directions is the Eigenvectors of A:

$$|A-\lambda I| = 0$$

$$\begin{vmatrix} \cos\theta-\lambda & -\sin\theta \\ \sin\theta & \cos\theta-\lambda \end{vmatrix} = 0$$

$$(\cos\theta-\lambda)^2 + \sin^2\theta = 0$$

$$\lambda^2 - 2\lambda\cos\theta + \underbrace{\cos^2\theta + \sin^2\theta}_{1} = 0$$

$a=1$ $\quad b=2\cos\theta$ $\quad c=1$

$$\underbrace{\lambda^2 - 2\lambda\cos\theta + 1}_{\text{can't factor so use the quadratic}} = 0$$

$$\lambda_{1,2} = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4(1)(1)}}{2(1)}$$

if $\theta \neq 0, 2\pi, 4\pi$, etc $(2n\pi)$ for $n=1,2,\ldots$

$\cos\theta < 1$, &

$4\cos^2\theta < 4$, so we have imaginary ~~roots~~, eigenvalues, giving us imaginary eigenvectors so there is no REAL principle direction if $\theta \neq 2n\pi$

if $\theta = 2n\pi$, n integer

$\lambda_{1,2} = \cos\theta$, so

$$(A-\lambda I)\vec{x} = 0 \Rightarrow \begin{bmatrix} \cos\theta-\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta-\cos\theta \end{bmatrix}\vec{x} = 0$$

but $\theta = 0$,

so

$$(A-\lambda I)\vec{x} \Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \vec{x} = \vec{0}$$

both x_1 & x_2 are free variables, infinite solutions, every direction is a principle direction !?

Reason

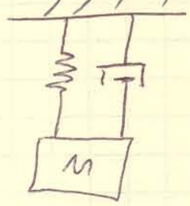
The whole point of the rotation matrix is to rotate every vector by the same θ, A principle direction is the direction in $\vec{x}$ for which **A** does not change that direction → Therefore if we have a rotation matrix, rotating every direction we cannot have a principle direction.

If $\theta = 0$, however, every $\vec{x}$ keeps its direction, so every direction is a principle direction ?!?

probably not the proper terminology

Back to Systems

1^{st} lets look @ our standard mass spring damper setup:



$$my'' + cy' + ky = 0$$

or

$$\underbrace{y'' = -\frac{c}{m}y' - \frac{k}{m}y}$$

we can convert this to two 1^{st} Order equations:

Let

$$y_1 = y$$
$$y_1' = y_2$$
$$y_2' = -\frac{k}{m}y_1 - \frac{c}{m}y_2$$

now I have two related 1^{st} Order Systems

Now let's set it up in matrix form:

$$\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\underline{\vec{y}' = A\vec{y}} \Rightarrow \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

looks similar to $\cancel{x' = ax}$ $y' = ay$
with sol $\cancel{x = ce^{a}}$ $y = ce^{at}$

lets use this as guide

$$y' = A\vec{y}$$

let $\vec{y} = \vec{c}\,e^{\lambda t}$ ← cnst vector

then $\vec{y}\,' = \lambda \vec{c}\,e^{\lambda t}$

giving us:

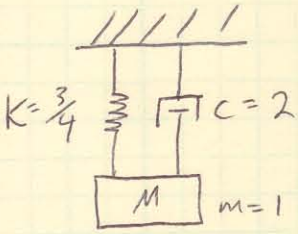
$$\vec{y}\,' = A\vec{y} \Rightarrow \lambda \vec{c}\,\cancel{e^{\lambda t}} = A\vec{c}\,\cancel{e^{\lambda t}}$$

$$\lambda \vec{c} = A\vec{c}$$ ← looks like an eigen problem

$$(A - \lambda I)\vec{c} = \vec{0}$$

λ ← eigen values; $\vec{c}$ ← eigen vectors

Lets do an example:



$$y'' + 2y' + 3/4 = 0$$

~~① Lets Find our A matrix:~~

① Lets set up our system:

let $y_1' = y_2$,

$y_2' = -2y_1 - 3/4$

$$\bar{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad \bar{y}\,' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix}$$

$$\vec{y}' = A\vec{y}$$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -.75 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$ check your equations!

$$\vec{y}' = A \quad \vec{y}$$

② Assume $\vec{y} = \vec{c}e^{\lambda t}$, so $\vec{y}' = \lambda\vec{c}e^{\lambda t}$

$$\lambda\vec{c}\cancel{e^{\lambda t}} = A\vec{c}\cancel{e^{\lambda t}}$$

$$\lambda\vec{c} = A\vec{c}$$

$$(A - \lambda I)\vec{c} = \vec{0}$$ ← This is our eigenvalue problem

③ Solve Eigenvalue problem!

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 0-\lambda & 1 \\ -.75 & -2-\lambda \end{vmatrix} = 0$$

$$\lambda^2 + 2\lambda + .75 = 0$$

$$(\lambda + .5)(\lambda + 1.5) = 0$$

$$\lambda = -.5, \ -1.5$$ ← two solutions

④ Find Eigenvectors ($\vec{c}$)

$\lambda_1 = -.5$

$$(A - \lambda I)\vec{c} = \vec{0}$$

$$\begin{bmatrix} .5 & 1 \\ -.75 & -1.5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \vec{0}$$ $R_2 = 1.5R_1 + R_2$

$$\begin{bmatrix} .5 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \vec{0}$$

free variable

$$.5c_1 + 1c_2 = 0$$

$$c_1 = -2c_2$$

$$c_2 = 1, \quad c_1 = -2$$

$$\vec{c} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$\lambda = -1.5$

$$\begin{bmatrix} 1.5 & 1 \\ -.75 & -.5 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \vec{0} \qquad R_2 = R_1 + 2R_2$$

$$\begin{bmatrix} 1.5 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \vec{0}$$

$$1.5c_1 + c_2 = \vec{0}$$

$$\vec{c} = \begin{bmatrix} 1 \\ -1.5 \end{bmatrix}$$

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(3) Put it all together,

we have $\lambda_1 = -.5, \ w/ \ \vec{c} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\lambda_2 = -1.5, \ w/ \vec{c} = \begin{bmatrix} 1 \\ -1.5 \end{bmatrix}$

what do we do with these?

we started with:

$$\underbrace{\vec{y} = \vec{c}\, e^{\lambda x}}$$

but we got two solutions!

what do we do with the two solutions?

Linear combo!

$$\vec{y} = C_1 \vec{c}_1 e^{\lambda_1 x} + C_2 \vec{c}_2 e^{\lambda_2 x}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = C_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-.5x} + C_2 \begin{bmatrix} 1 \\ -1.5 \end{bmatrix} e^{-1.5x}$$

$$y_1 = -2C_1 e^{-.5x} + C_2 e^{-1.5x}$$
$$y_2 = C_1 e^{-.5x} + -1.5 C_2 e^{-1.5x}$$

? does $\underbrace{y_2 = y_1'}_{\text{pre condition}}$ ✓

$$y_{original} = y_1 = -2C_1 e^{-.5x} + C_2 e^{-1.5x}$$
$$y'_{original} = y_2 = C_1 e^{-.5x} + -1.5 C_2 e^{-1.5x}$$

Doing the problem this way, we get both y & y'!

Bonus:

What do the Eigenvalues tell us about the mass spring system?

BIG BONUS

Solve the following mass spring system with Eigenvalues & vectors → Check work either with Laplace or other technique:

$$my'' + cy' + ky = 0$$

$m = 1$

$c = 4$

$k = 5$