Curves, Tangents, Arclength

REVIEW: Dot Product sec 9.2
Cross Product sec 9.3
General Vector Prop sec 9.1

Bonus: pg 370 #4 Read & Do entire Problem
pg 376 #28
pg 383 #11

Let $\underset{\uparrow}{\vec{r}(t)} = \underbrace{< x(t), y(t), z(t) >}_{\text{Cartesian coordinates}} = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$

parametric representation of a curve

unit vector along x-axis $<1,0,0>$

unit vector along y axis $<0,1,0>$

unit vector along z-axis $<0,0,1>$

→ puts our cartesian coordinates in terms of another variable
ex: position vs time, velocity vs time, acceleration vs time

Lets Look at the parametric representations of a few common curves!

Circle

$x^2 + y^2 = r^2$

whenever we have an x^2 & y^2 try to get to the point where $(\ )^2 + (\ )^2 = 1$
then use trig! $\cos^2 t + \sin^2 t = 1$

in This example!

$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$

$\cos^2 t + \sin^2 t = 1$

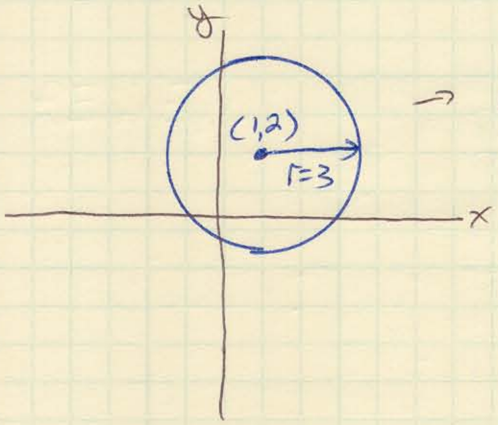
$\left(\frac{x}{r}\right)^2 = \cos^2 t \Rightarrow x = r\cos t$

$\left(\frac{y}{r}\right)^2 = \sin^2 t \qquad y = r\sin t$

??

what happens if I & changes let $\left(\frac{x}{r}\right)^2 = \sin^2 t$ & $\left(\frac{y}{r}\right)^2 = \cos^2 t$? in detail!

Offset circle



→ whats the equation?

if center was (0, 0)

$x^2 + y^2 = r^2$

since center not origin, shift

$(x-1)^2 + (y-2)^2 = 3^2$

Now get to form:

$(\quad)^2 + (\quad)^2 = 1$

$\left(\frac{x-1}{3}\right)^2 + \left(\frac{y-2}{3}\right)^2 = 1$

$\quad \| \qquad\qquad \|$

$\cos^2 t \qquad \sin^2 t$

$x = 1 + 3\cos t, \quad y = 2 + 3\sin t$

Ellipse

$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$

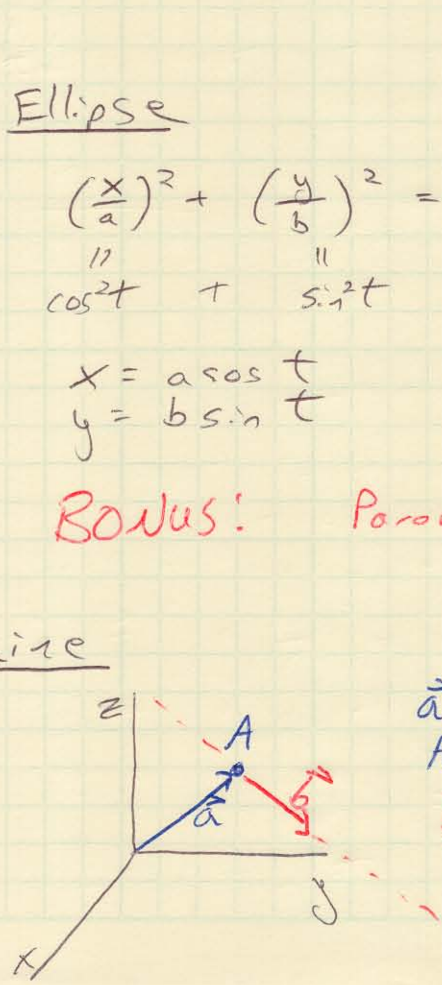
$\quad \| \qquad\qquad \|$

$\cos^2 t \quad + \quad \sin^2 t \quad = 1$

$x = a\cos t$
$y = b\sin t$

BONUS! Parameterize! $16x^2 + 9y^2 = 144$

Line



$\vec{a}$ = vector to start point

A = start point = (x_0, y_0, z_0)

$\vec{b}$ = vector in direction of line (think of as the velocity vector)

$\vec{r}(t) = \vec{a} + t \cdot \vec{b}$

$$\vec{r}(t) = \vec{a} + t \cdot \vec{b}$$

start point ↑   ↑ time of travel   ↖ velocity

<u>Other Fun:</u>

given $\vec{r}(t) = \langle 2 + r\cos 4t,\ 6 + r\sin 4t,\ 2t \rangle$

Find the non parametric representation:

$$\left.\begin{aligned} 2 + r\cos 4t &= x \\ 6 + r\sin 4t &= y \end{aligned}\right\}$$ can we work this into $(\ \ )^2 + (\ \ )^2 = 1$?

$$r\cos 4t = x - 2$$
$$r\sin 4t = y - 2$$

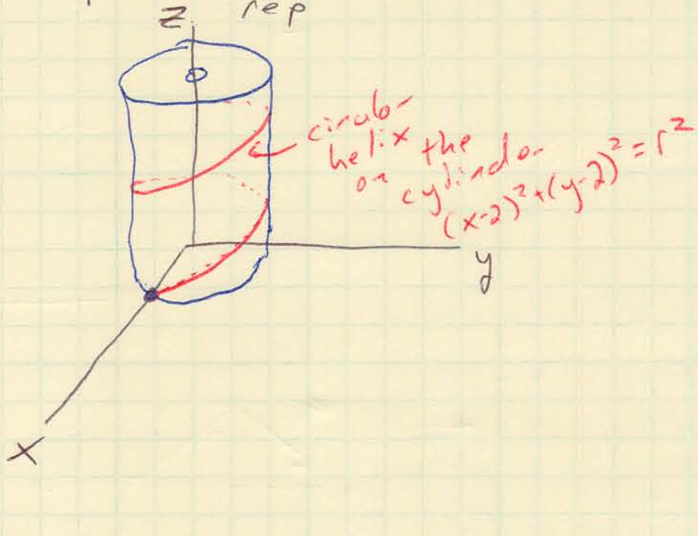
$$\left.\begin{aligned} \cos 4t &= \frac{x-2}{r} \\ \sin 4t &= \frac{y-2}{r} \end{aligned}\right\}$$ what about the 4ts?

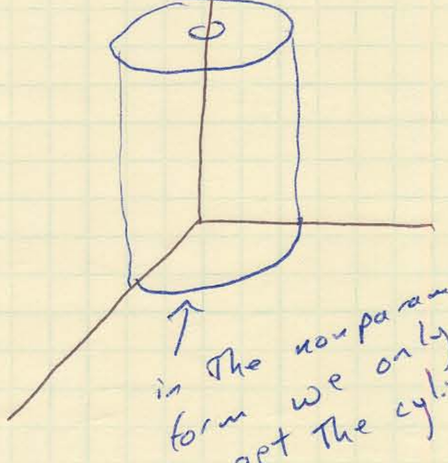
$$\cos^2 4t + \sin^2 4t = 1$$

$$\left(\frac{x-2}{r}\right)^2 + \left(\frac{y-2}{r}\right)^2 = 1$$

$$(x-2)^2 + (y-2)^2 = r^2 \quad , -\infty \le z \le \infty$$

parametric rep   non parametric





in the nonparametric form we only get the cylinder

Tangent to a Curve

use same method used for finding derivative:

$$F'(x) = \lim_{\Delta x \to 0} \frac{F(x+\Delta x) - F(x)}{\Delta x}$$

tangent

but in vector form:

$$\vec{r}\,'(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[\vec{r}(t+\Delta t) - \vec{r}(t)\right]$$

but $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

so

$$\vec{r}\,'(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[\langle x(t+\Delta t) - x(t), y(t+\Delta t) - y(t), z(t+\Delta t) - z(t) \rangle\right]$$

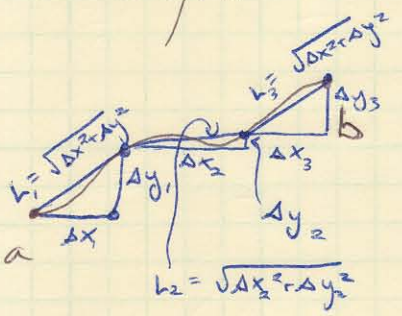
$$= \langle x'(t), y'(t), z'(t) \rangle$$

unit tangent:

$$\vec{u} = \frac{1}{|\vec{r}\,'|}\vec{r}\,'$$

Arclength

Find length of curve C,



$$\ell = \Sigma L$$

but we know

$$x'(t) \approx \frac{\Delta x}{\Delta t} \quad \& \quad y'(t) \approx \frac{\Delta y}{\Delta t}$$

so $\Delta x \approx x'(t)\Delta t$, $\Delta y \approx y'(t)\Delta t$

$$L = \sqrt{\Delta x^2 + \Delta y^2}$$

$$= \sqrt{(x'(t)\Delta t)^2 + (y'(t)\Delta t)^2}$$

$$= \sqrt{x'(t)^2 + y'(t)^2}\;\Delta t$$

$$\ell \approx \sum_{i=1}^{n} \sqrt{x'(t)^2 + y'(t)^2}\;\Delta t$$

Leading to

$$l = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

$$\vec{r}(t) = \langle x(t), y(t) \rangle$$

$$l = \int_a^b |\vec{r}\,'(t)|\, dt$$

or

$$l = \int_a^b \sqrt{\vec{r}\,'(t) \cdot \vec{r}\,'(t)}\, dt$$

arclength as a function of t

(how far did you travel after t seconds)

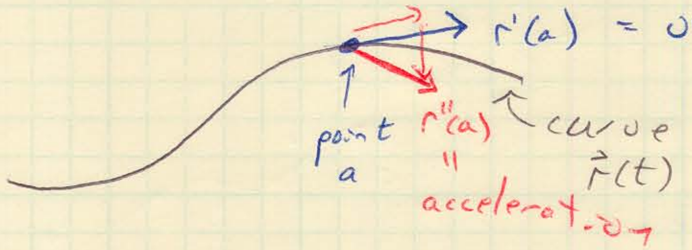
$$s(t) = \int_a^t \sqrt{r'(t) \cdot r'(t)}\, dt$$

Velocity & Acceleration

if $\vec{r}(t)$ is position, then

$\vec{r}\,'(t)$ is velocity $= \vec{v}(t)$

$\vec{r}\,''(t)$ is acceleration $= \vec{v}\,'(t) = \vec{a}(t)$



velocity is always <u>tangent</u> to curve

Acceleration consists of normal component & tangential component

$$\vec{a} = \vec{a}_{tan} + \vec{a}_{norm}$$

$$\vec{a}_{tan} = \frac{\vec{a} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}\, \vec{v} = \overbrace{\underbrace{(\vec{a} \cdot \vec{v})}_{\text{component of } \vec{a} \text{ in } \vec{v} \text{ direction}}}^{\text{scalar}} \overbrace{\underbrace{\left(\frac{\vec{v}}{\vec{v} \cdot \vec{v}}\right)}_{\text{vector}}}^{\text{unit tangent vector}}$$

$$\vec{a}_{norm} = \vec{a} - \vec{a}_{tan}$$