Gradient of a Scalar Field
Directional Derivative

First lets define gradient:

$$\text{grad } F(x,y,z) = \nabla F = \left[\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right] = \frac{\partial F}{\partial x}\vec{i} + \frac{\partial F}{\partial y}\vec{j} + \frac{\partial F}{\partial z}\vec{k}$$

ex: if $F(x,y,z) = 3x^2 y z^3 + 4xz$

then $\nabla F = \langle \underbrace{6xyz^3 + 4z}_{\frac{\partial F}{\partial x}}, \underbrace{3x^2 z^3}_{\frac{\partial F}{\partial y}}, \underbrace{9x^2 y z^2 + 4x}_{\frac{\partial F}{\partial z}} \rangle$

nabla

What do we use gradients for?

1. Finding directional derivatives
2. Obtaining surface normals
3. deriving vector fields from scalar fields

What does it tell us?

BONUS

if ∇f exists ($\boxed{\text{What would cause it not to exist?}}$), & it is not $\vec{0}$ at some point P, then $\nabla F(P)$ has the direction of maximum increase of **F** @ P.

grad function at point P

1. Finding Directional Derivatives:

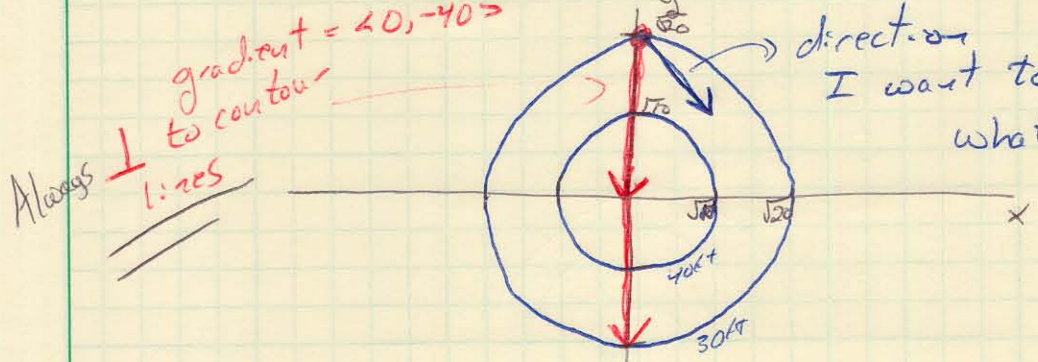
Lets think of a hilltop defined by

$F(x,y) = 50 - x^2 - y^2$ m

Now, lets say we didn't want to change altitude at all? What direction would we head in?

→ Follow the contour, so $\langle 1, 0\rangle$ or $\langle -1, 0\rangle$ direction

What if I wanted to head in the $\langle 10, -10\rangle$ direction? what is ~~my~~ the slope?



Gradient gives max slope, so we need to find the component of our direction pointing in direction of gradient to find how fast we are rising!

$$D_{\vec{a}} F = \underbrace{\frac{1}{|\vec{a}|}\vec{a}}\cdot \nabla F$$

gives us the component of $\vec{a}$ in direction of gradient

so for our vector

$$\text{slope} = D_{\langle 10,-10\rangle} F = \frac{1}{\sqrt{10^2+10^2}} \langle 10, -10\rangle \cdot \langle 0, \cancel{-40}\,{-2\sqrt{20}}\rangle$$

$$= \frac{\cancel{400}\; 20\sqrt{20}}{\sqrt{200}} = \underbrace{\frac{\cancel{40}\; 2\sqrt{20}}{\sqrt{2}}}_{\text{scalar quantity}}$$

Can we plot it? ~~How do we~~

What is a good technique for representing a 3-D object in 2-D? Think of your maps → Level curves

$$z = 50 - x^2 - y^2$$

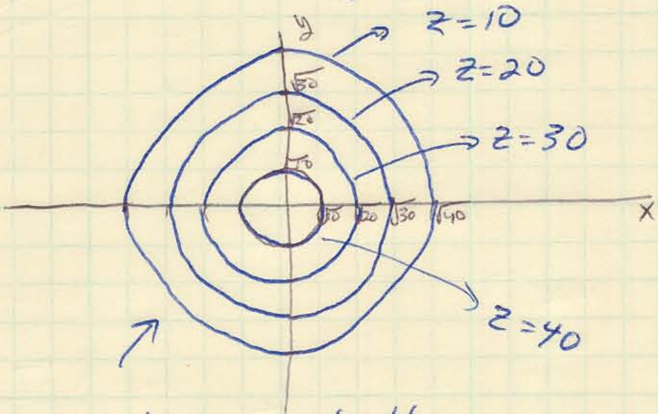
to use level curves, look at constant values of the Function:

By choosing cnst z values we can get a contour plot

$$\boxed{40} = 50 - x^2 - y^2 \rightarrow x^2 + y^2 = 10 \leftarrow \text{plot this}$$

$$\boxed{30} = 50 - x^2 - y^2 \rightarrow x^2 + y^2 = 20$$

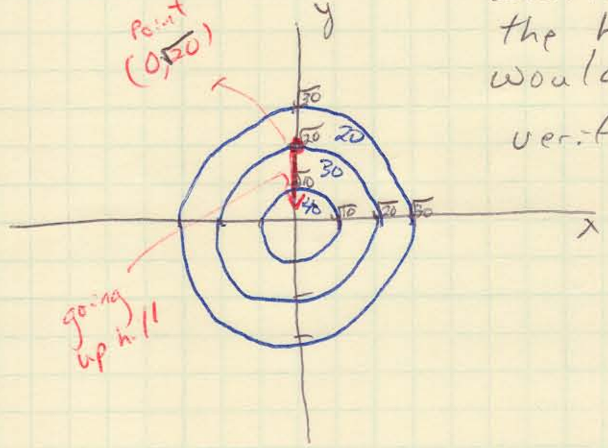
$$\boxed{20} = 50 - x^2 - y^2 \rightarrow x^2 + y^2 = 30$$



looks like a hill

Ok so now we have our plot, imagine standing at the point $(0,\sqrt{20})$, if you wanted to go directly up the hill which direction would you go? $\underline{\langle 0,-1\rangle}$

verify with gradient:



$$\nabla F(\underset{x}{0},\underset{y}{\sqrt{20}}) = -2x\vec{i} - 2y\vec{j}$$

$$= \cancel{0\vec{i} - 40\vec{j} = \langle 0,-40\rangle}$$

$$= 0\vec{i} - 2\sqrt{20}\vec{j} = \langle 0,-2\sqrt{20}\rangle$$

Same direction!

2. Gradient as a Surface Normal

just as we have level curves to represent a 3-D image in 2-D, we can have level surfaces to represent a $\underbrace{\text{4-D image in 3-D}}$

think of temperature in a uniform heated ball as it cools, (single point in time)

$$T = 50 - \underbrace{x^2 - y^2 - z^2}$$

temp at center

as we move away from the center, the temp is less

if we look at certain cnst temperatures,

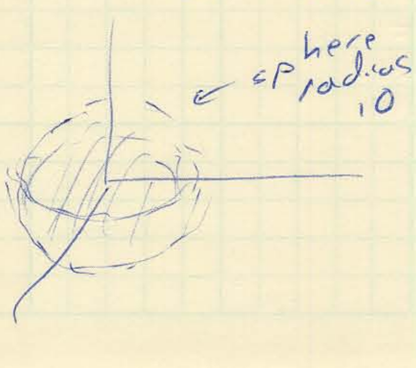
$T = 40$
$T = 30$
$T = 20$ } we get $\underbrace{c = f(x, y, z)}_{\text{level surface}}$

look @ $T = 40$

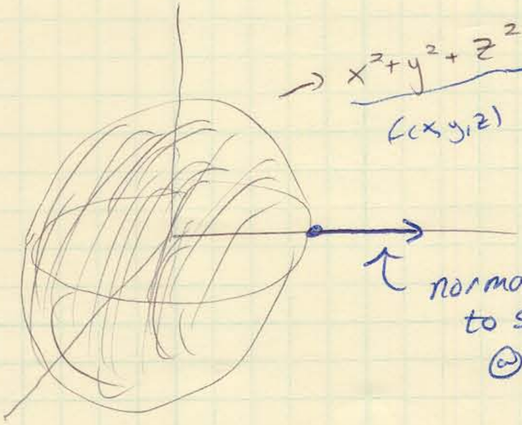
$$40 = 50 - x^2 - y^2 - z^2$$

$$\Rightarrow \underbrace{x^2 + y^2 + z^2 = 10}$$

the level surface in our ball with temperature $= 40$, is a sphere of radius 10



Just as with level curves, gradient is normal $\perp$ to a level surface.



$\rightarrow x^2+y^2+z^2=10$

$f(x,y,z)=c$

normal to surface @ $(0,\sqrt{10},0)$

Find surface normal @ $(0,\sqrt{10},0)$ of The level surface $x^2+y^2+z^2=10$:

$\vec{n} = \nabla f(p)$

grad, level surface function, at point p

$\nabla f(x,y,z) = \langle 2x, 2y, 2z \rangle$ @ $\langle 0,\sqrt{10},0 \rangle$

$= \langle 0, 2\sqrt{10}, 0 \rangle$

Caution: is that The direction of maximum temp increase??

BONUS: Explain why or why not that for our function $T = 50-x^2-y^2-z^2$, the gradient of $\underbrace{x^2+y^2+z^2=10}_{\text{a level surface}}$ is in The direction of max temp increase!

3. Vector Fields that are gradients of scalar fields

Lets look @ Potential Energy due to gravity:

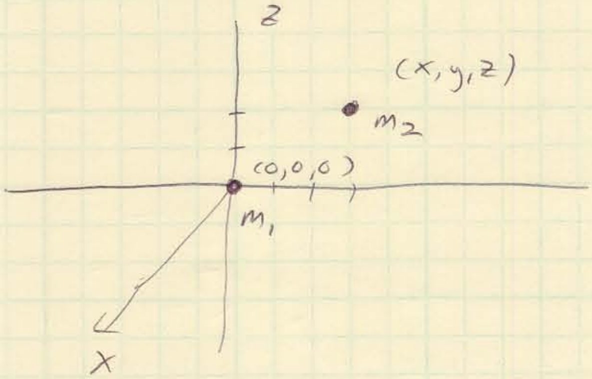
$$PE = -G\,\frac{m_1 m_2}{R} \quad \text{on earth} = m \cdot g \cdot h$$

$M_{earth} \cdot G$

mass of object

Radial distance R

two planets:



$$PE = -G \cdot \frac{m_1 m_2}{\sqrt{x^2+y^2+z^2}} = \text{Potential function}$$

but what is the Forces acting on the planets?

<u>Gravitational Field</u>

Gravitational Field = gradient of the potential function

$$GF = \nabla\left(-G \cdot \frac{m_1 m_2}{\sqrt{x^2+y^2+z^2}}\right)$$

Any vector field that is a gradient of a potential Function is called <u>conservative</u> → energy is neither lost nor gained!

We will discuss more in next block!