Divergence

useful link: www.math.umn.edu/~nykamp/m2374/readings/divcurl/

Previous: Gradient $\rightarrow \nabla F(x,y,z) = \langle \frac{\partial}{\partial x} F, \frac{\partial}{\partial y} F, \frac{\partial}{\partial z} F \rangle$

$\hookrightarrow$ nabla $= \nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$

so ∇F

Gradient: vector · scalor = vector

takes scalor & gets vector

Extending the ∇ vector concept, we define divergence:

$$\underbrace{\vec{\nabla} \cdot \vec{F}}_{} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle F_x, F_y, F_z \rangle$$

dot product only works between vectors so F is a vector valued function

$$= \underbrace{\frac{\partial}{\partial x} F_x + \frac{\partial}{\partial y} F_y + \frac{\partial}{\partial z} F_z}_{}$$

divergence Takes a vector & gives us a scalar

so

Gradient takes a scalor function & creates a vector

Divergence takes a ~~scalor~~ vector function & creates a scalor

but what does it mean?

First, example:

$$\vec{V} = \langle 3x^2y, 2z, 3yxz^3 \rangle$$

$$\vec{\nabla} \cdot \vec{V} = \frac{\partial}{\partial x}(3x^2y) + \frac{\partial}{\partial y}(2z) + \frac{\partial}{\partial z}(3yxz^3)$$

$$= 6xy + 0 + 9yxz^2 = \underline{\underline{6xy + 9yxz^2}}$$

The Divergence of $\vec{V}$ @ point x, y, z

What Does it Mean?

<u>Gradient</u> gives maximum rate of change

what is <u>Divergence</u>?

<u>Fluid Flow</u>

<u>water & oil</u>

very low compressibility, essentially means ρ (density) is constant

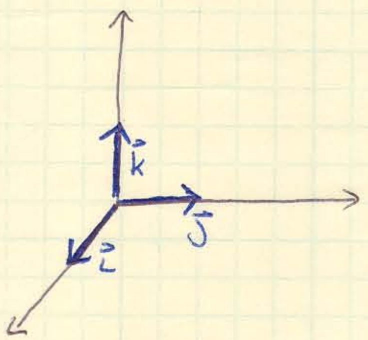
<u>gasses & vapor</u>

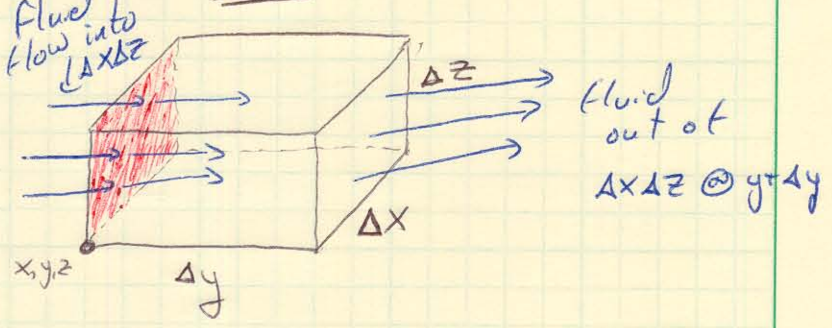
high compressibility density can change with position & time

$PV = nRT$

Lets look @ Fluid Flow through a box, @ the point (x, y, z), with sides $\Delta x, \Delta y, \Delta z$

<u>Box B</u>





We are going to look @ the change in mass included in B by considering the Flux across the boundaryies

amount of material per unit area per unit time $\rightarrow \frac{kg}{m^2 s}$

First lets look @ $\Delta x \Delta z$ Face
(the red face)

The Fluid velocity Field is defined as

$\vec{v}$ = Fluid velocity Field

From this we can get the Flux:

$$\vec{u} = \underbrace{\rho}_{\text{density}} \cdot \underbrace{\vec{v}}_{\text{velocity}}$$

$$\frac{kg}{m^3} \quad \frac{m}{s} = \frac{kg}{m^2 s} \Big\} \text{Flux}$$

Since we are looking at the $\Delta x \Delta z$ face, does any fluid flow in the $\vec{i}$ or $\vec{k}$ directions pass through the face?

No, $\vec{i}$ & $\vec{k}$ flows run parallel

So, mass entering the $\Delta x \Delta z$ face is:

$$(\rho v_2)_y \, \Delta x \Delta z \Delta t = (u_2)_y \, \Delta x \Delta z \Delta t$$

refers to left face

flux @ y in $\vec{j}$ comp

Mass entering: $(u_2)_y \, \Delta x \Delta z \Delta t$

what about mass leaving?

mass leaving $\underbrace{(u_2)_{y+\Delta y}}_{\text{flux @ } y+\Delta y \text{ in } \vec{j} \text{ comp}} \Delta x \Delta z \Delta t$

Lets define

$$\Delta u_2 = (u_2)_{y+\Delta y} - (u_2)_y$$

& look at mass entering & leaving $\Delta x \Delta z$ Faces

$$\underbrace{\Delta u_2 \Delta x \Delta z \Delta t}_{\text{mass entering - mass leaving}} = \underbrace{\frac{\Delta u_2}{\Delta y} \overbrace{\Delta V}^{\text{Volume of box}} \Delta t}_{\text{approx loss of mass}}$$

how??

ΔV

$\Delta u_2 \Delta x \Delta z = \Delta u_2 \frac{\Delta x \Delta z \Delta y}{\Delta y}$

flux $\dfrac{\frac{kg}{m^2 s}}{m} \cdot m^3 \cdot s = kg$

(flux, distance, Volume, time)

We can derive similar expressions for the other faces, giving us:

$$\left(\frac{\Delta u_1}{\Delta x} + \frac{\Delta u_2}{\Delta y} + \frac{\Delta u_3}{\Delta z}\right) \overbrace{\Delta V}^{\text{Volume of box}} \Delta t$$

change in mass during time Δt

But, for there to be a loss of mass inside the box over some Δt, there must be a change in density, so change in mass over Δt equivalent to

$$-\frac{\partial \rho}{\partial t} \Delta V \Delta t$$

time rate of change of density

$$\frac{\frac{kg}{m^3}}{s} \cdot m^3 \cdot s = kg$$

Equating are two ~~rates of~~ changes in mass:

$$\underbrace{\left(\frac{\Delta u_1}{\Delta x} + \frac{\Delta u_2}{\Delta y} + \frac{\Delta u_3}{\Delta z}\right)}_{\lim_{\Delta x, \Delta y, \Delta z \to 0}} \cancel{\Delta V \Delta t} = -\frac{\partial \rho}{\partial t} \cancel{\Delta V \Delta t}$$

$$\mathrm{Div}(\vec{u}) = -\frac{\partial \rho}{\partial t}$$

the divergence of flux equals the rate of change in density or

$$\vec{\nabla}\cdot\vec{\omega} + \frac{\partial \rho}{\partial t} = 0$$

or

$$\nabla\cdot\rho\vec{v} + \frac{\partial \rho}{\partial t} = 0$$

if Fluid is incompressible, Then $\frac{\partial \rho}{\partial t} = 0$
and

$\vec{\nabla}\cdot\vec{v} = 0$ → condition for incompressibility!!

Other uses of Divergence:

$\nabla\cdot\vec{B} = 0$ — magnetic flux density

$\nabla\cdot\vec{D} = \rho$ — electric flux density

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what do These say?

$\nabla\cdot\vec{B} = 0$ } says whatever comes into our little box goes out!

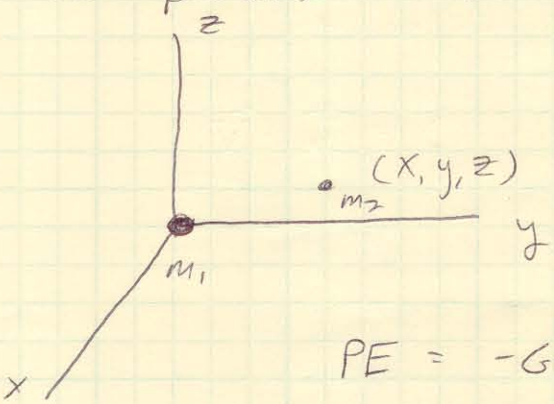
$\nabla\cdot\vec{D} = \rho$ } says if we have charge inside our box, Then we have more going out than in

Gravitational Fields

Potential Energy due to Gravity:

$PE = -G \frac{m_1 m_2}{R}$ on earth: $m \cdot g \cdot h$

two planets (earth & moon)



$$PE = -G \cdot \frac{m_1 m_2}{\sqrt{x^2+y^2+z^2}} = \text{Potential Function}$$

but what are the Forces?

A Force Field is obtained by taking the gradient of a Potential Energy Function

Find the gravitational Force Field of the PE function

$$\vec{G} = \vec{\nabla} PE = \vec{\nabla}\left(-G \frac{m_1 m_2}{\sqrt{x^2+y^2+z^2}}\right)$$

Any vector field that is the gradient of a Potential Function is called

conservative

$\vec{V} = \vec{\nabla} F$

→ energy is neither lost nor gained