Line Integrals

Standard integral interpreted as a line integral:

x is our path

f(x) is our value along the path

dx is the incremental distance of path

so we integrate:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_0^n f(x_n)\,\Delta x$$

$$\Delta x = \frac{b-a}{n}$$

Often times our path is not straight:

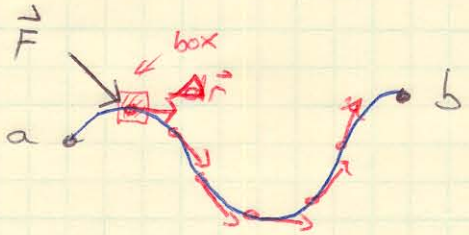
work:

path = $\vec{r}(t)$

$f(x, y, z)$ = force

Work = $F \cdot d$

for a straight line distance, what if we have a curved path as above?



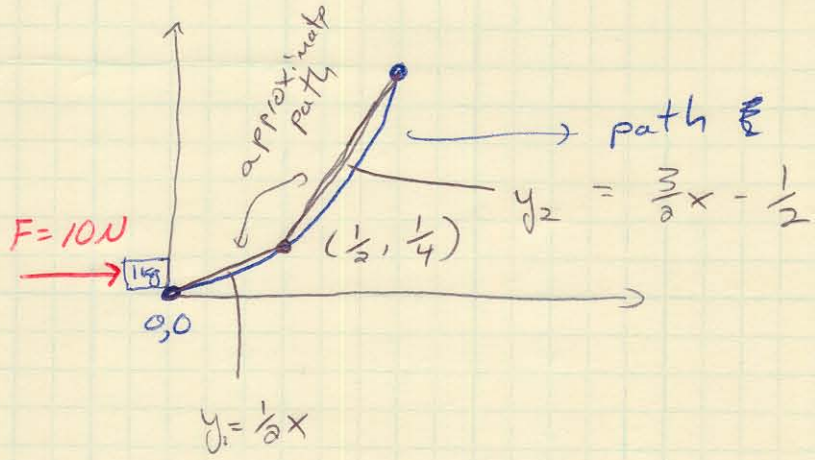
break path up into segments

so total Force:

$$W_{tot} = \sum_0^n \vec{F} \cdot \Delta\vec{r}$$

Ex 1: How much work is required to Move the 1kg Box along the path ~~$y = \frac{1}{2}x$~~ $y = x^2$, from (0,0) to (1,1). The Force is $\langle 10N, 0 \rangle$

1st Break it into two pieces:



$$W_1 = \vec{F} \cdot \vec{d}_1 = \langle 10, 0 \rangle \cdot \langle \tfrac{1}{2}, \tfrac{1}{4} \rangle$$
$$= 5 \text{ Nm}$$

$$W_2 = \vec{F} \cdot \vec{d}_2 = \langle 10, 0 \rangle \cdot \langle \tfrac{1}{2}, \tfrac{3}{4} \rangle$$
$$= 5 \text{ Nm}$$

$$W_{tot} = 10 \text{ Nm}$$

~~Bon~~ Bonus: Find the W_{tot} by breaking it into 4 segments

What if we let the $\Delta \vec{r}(d)$ go to 0?

then

$$W = \int \vec{F}(\vec{r}) \cdot d\vec{r}$$

infinite sum — Force — incremental distance in direction of path

Ex 2: Same problem, but using

$W = \int \vec{F} \cdot d\vec{r}$

$W = \int \langle 10, 0 \rangle \cdot \underbrace{\langle \quad \rangle}$

? what is $d\vec{r}$

well it's in the direction of $\vec{r}$, so it is tangent to $\vec{r}$ ... what's tangent to $\vec{r}$? $\vec{r}\,'(t)$

How do we get the incremental distance though? travel in direction of $\vec{r}\,'(t)$ for a very short period of time, dt

$$\boxed{d\vec{r} = r'(t)\,dt}$$

$W = \int \langle 10, 0 \rangle \cdot \underbrace{\vec{r}\,'(t)}\, dt$

Great, what's $\vec{r}\,'(t)$ or $\vec{r}(t)$?

Need to parametrize curve $y = x^2$

$x(t) = t$

$y(t) = t^2$

$\vec{r}(t) = \langle t, t^2 \rangle$

Finally, $r'(t) = \langle 1, 2t \rangle$

$$W = \int_a^b \underbrace{\langle 10, 0 \rangle}_{\vec{F}} \cdot \underbrace{\langle 1, 2t \rangle \, dt}_{d\vec{r}}$$

$$= \int_0^1 10 \, dt = \underline{10 \text{ Nm}}$$

Same answer as before

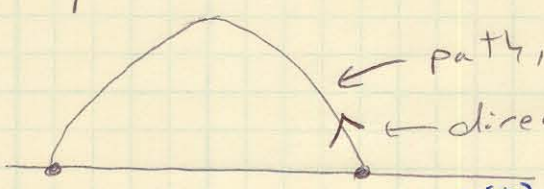
$\boxed{\text{Bonus Why?}}$

Ex: $\vec{F} = \langle x^2, y^2, 0 \rangle$

C = semicircle from $(2,0)$ to $(-2,0)$, $y \geq 0$

curve, or path

plot it!



Need to parameterize!

$$\vec{r}(t) = \langle \underbrace{2\cos t}_{x(t)}, \underbrace{2\sin t}_{y(t)} \rangle$$
$0 < t < \pi$

is direction correct?
is start point correct?

so

$$W = \int_{t_0}^{t_F} \vec{F}(\vec{r}) \cdot d\vec{r}$$

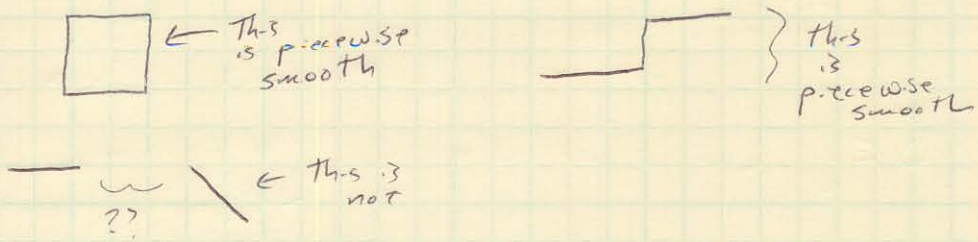
$$= \int_0^{\pi} \langle (x(t))^2, (y(t))^2, 0 \rangle \cdot r'(t) \, dt$$

$$= \int_0^{\pi} \langle 4\cos^2 t, 4\sin^2 t, 0 \rangle \cdot \langle -2\sin t, 2\cos t, 0 \rangle \, dt$$

$$= \int_0^{\pi} -8\cos^2 t \sin t + 8\sin^2 t \cos t \, dt$$

$$= \underbrace{\int_0^{\pi} -8\cos^2 t \sin t \, dt}_{\text{need to use u-sub on}} + \underbrace{\int_0^{\pi} 8 \sin^2 t \cos t \, dt}_{\text{need to use u-sub on}}$$

Note: General Assumption is that the curve is piecewise smooth



Another Form:

$$\int_C \vec{F} \cdot d\vec{r} = \underbrace{\int_C F_1 dx + F_2 dy + F_3 dz}_{\text{this is harder}}$$

I have never used this technique

Theorems:

1: Any representation of the curve, C, that gives the same positive direction also yields the same value for the ~~our~~ line integral.

huh? → Essentially, regardless of how you represent the path, as long as it is the same path and is going in the same + direction, you get the same answer.

2. The line integral <u>generally</u> depends not only on $\vec{F}$ & the endpoints, but also the path.

Bonus: When doesn't it depend on the path?

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