Lesson 7 Laplace Transforms
Sec 6.1
Problems → pg 226
2, 5, 6, 8,
15, 29, 30,
40

Let's Start with a review of Partial Fractions:

in grade school we learned to put two fractions over a common denominator & combine, i.e.:

$$\frac{3s}{s+4} + \frac{5}{s+6} \Rightarrow \frac{3s(s+6)+5(s+4)}{(s+4)(s+6)}$$

$$\Rightarrow \frac{3s^2+23s+20}{(s+4)(s+6)}$$

Partial Fractions allow us to do the opposite, i.e., given a single fraction, break it up into smaller pieces i.e.

$$\frac{20}{(s-1)(s+4)} = \frac{A}{s-1} + \frac{B}{s+4}$$

to Find A & B, need two equations, so choose two values ~~of~~ for s & solve

↳ If we choose wisely, we can make the work easier!

Mult by through $(s-1)$

$$\frac{20}{s+4} = A + \frac{B(s-1)}{s+4}$$

Let $s=1$

$$\frac{20}{5} = A + \cancel{\frac{B(1-1)}{s+4}}^{\,0}$$

$$A = 4$$

Mult by $(s+4)$

$$\frac{20}{s-1} = \frac{A(s+4)}{(s-1)} + B$$

Let $s=-4$

$$\frac{20}{-5} = \cancel{\frac{A(-4+4)}{(s-1)}}^{\,0} + B$$

$$B = -4$$

So $\frac{20}{(s-1)(s+4)} = \frac{-4}{(s+4)} + \frac{4}{(s-1)}$

For More Techniques Review:

http://www.mathsrevision.net/alevel/pages.php?page=94

Why use Laplace Transforms to solve Differential Eqs?

1. Easy to solve Initial Value Problems (IVP)
2. Solves Differential Eqs Algebraically
3. Handles Discontinuities Easily!

What is a Laplace Transform?

$$F(s) = \mathcal{L}(f(t)) = \underbrace{\int_0^\infty e^{-st} f(t)\, dt}_{\text{what the transform is}}$$

the transform of a time domain function $f(t)$

Laplace Transform

original function

Example:

Find the Laplace transform of t.

$$F(s) = \mathcal{L}\{\underset{f(t)}{t}\} = \underbrace{\int_0^\infty e^{-st} \cdot \overset{f(t)}{t}\, dt}_{\text{This is the work we must do}}$$

$$\int_0^\infty e^{-st}\, t \; dt =$$

Use integration by parts
pg 9 in FE, # 6

$$\int u\, dv = uv - \int v\, du$$

$$\boxed{\begin{array}{lll} dv = e^{-st} & \rightarrow & v = \frac{-e^{-st}}{s} \\ u = t & \rightarrow & du = 1 \end{array}}$$

$$\left[\frac{-e^{-st} \cdot t}{s} - \int \frac{-e^{-st}}{s}\, dt \right]_0^\infty$$

$$\frac{-e^{-st} t}{s}\Big|_0^\infty - \frac{+e^{-st}}{s^2}\Big|_0^\infty$$

$$0 - 0 \quad - \left(0 - \frac{1}{s^2} \right) = \underline{\underline{1/s^2 = F(s)}}$$

Linearity of Transform:
since integrals are Linear & the transform is nothing more than an integral, Laplace Transforms are also Linear:

$$\mathcal{L}\{aF(t) + b\,g(t)\} = a\,\mathcal{L}\{F(t)\} + b\,\{\mathcal{L}\{g(t)\}\}$$

$$= a F(s) + b G(s)$$

~~S = Sh~~

Are we going to Do These Integrals??
that doesn't seem easy??
Mostly NO!
We use Tables
see pg 224 in text

Example:

Find Laplace Transform of

$$t^4 + 4\sin 3t + e^{5t}t^2$$

$\mathcal{L}\{t^4 + 4\sin 3t + e^{5t}t^2\} \Rightarrow$ using linearity

$$\underbrace{\mathcal{L}\{t^{4}\}}_{\text{in table}} + 4\underbrace{\mathcal{L}\{\sin 3t\}}_{\text{in table}} + \underbrace{\mathcal{L}\{e^{5t}t^2\}}_{\text{Not in table?}}$$

$4 \to n$, $3 \to \omega$

Now, use table

$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$ $\qquad$ $\mathcal{L}\{\sin \omega t\} = \frac{\omega}{s^2+\omega^2}$

$\Downarrow$ $\qquad$ $\Downarrow$

$\frac{4!}{s^5} = \frac{24}{s^5}$ $\qquad$ $4 \cdot \frac{3}{s^2+3^2}$

$= \frac{12}{s^2+9}$

Next page

S-Shifting Theorem:

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

&

$$e^{at}f(t) = \mathcal{L}^{-1}\{F(s-a)\}$$

so we have $\mathcal{L}\{e^{\overset{\to a}{5}t}\overset{\to F(t)}{t^2}\}$

Notice we ignored the e^{5t} in finding $F(s)$

so lets first find $F(s)$

$$F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{t^2\} = \frac{2}{s^3}$$

directly from table

so $F(s) = \frac{2}{s^3}$ & $\mathcal{L}\{e^{\overset{a}{5}t}f(t)\} = F(s-5)$

if $F(s) = \frac{2}{s^3}$ then $F(s-5) = \frac{2}{(s-5)^3}$

& putting all the pieces back together:

$$\mathcal{L}\{t^4 + 4\sin 3t + e^{5t}t^2\} = \frac{24}{s^5} + \frac{12}{s^2+9} + \frac{2}{(s-5)^3}$$

What if I want to go back?

Inverse Laplace Transform

Another Integral?? → Use the table, backwards.

(this may require Algebra!)

Example

Denotes inverse transform

$$\mathcal{L}^{-1}\left\{ \frac{3}{s^4} + \frac{s-5}{(s-5)^2+9^2} + \frac{15}{s^2+4s+29} \right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{3}{s^4}\right\} + \mathcal{L}^{-1}\left\{\frac{s-5}{(s-5)^2+9}\right\} + \mathcal{L}^{-1}\left\{\frac{15}{s^2+4s+29}\right\}$$

Lets start with

$\mathcal{L}^{-1}\left\{\frac{3}{s^4}\right\} \rightarrow$ what does it look like?

table 6.1, #4, but not exactly

$$\frac{3}{s^4} \approx \frac{n!}{s^{n+1}}$$

what must $n = ?$

$4 = n+1$, so

$n = 3$

$$\frac{3\cdot 2\cdot 1}{s^4} \Rightarrow t^3$$

$3 \neq n!$

$3 = n!/2$

so

$$\mathcal{L}^{-1}\left\{\frac{3}{s^4}\right\} = \frac{t^3}{2}$$

Next: $\mathcal{L}^{-1}\left\{\frac{s-5}{(s-5)^2+9}\right\} \rightarrow$ what does it look like?

table 6.1, # 11

$$\frac{s-5}{(s-5)^2+9} = \frac{s-a}{(s-a)^2+\omega^2}$$

$a = 5$, $\omega = 3$

$$\Downarrow$$

$$e^{at}\cos\omega t$$

$$\mathcal{L}^{-1}\left\{\frac{s-5}{(s-5)^2+9}\right\} = e^{\overset{a}{5}t}\cos \overset{\omega}{3}t$$

Finally: $\mathcal{L}^{-1}\left\{\frac{15}{s^2+4s+29}\right\}$ → what does it look like?

Nothing

Can we modify it?

$$\frac{15}{s^2+4s+29}$$

↳ can't Factor

- 1st step try factoring, then break up w/ partial Fractions

- if that doesn't work, complete square

$$\underbrace{s^2+4s}+29$$

↓

check: $(s+2)^2-4 \quad +29$

$s^2+2s+2s+4-4$

s^2+4s ✓

$(s+2)^2+25$ → Now we have $\frac{15}{(s+2)^2+25}$ → Now it looks similar to #12, but the const. is off.

Fix by multiplying by

$$\frac{5}{5}\cdot\frac{15}{(s+2)^2+25} \Rightarrow \frac{15}{5}\,\frac{5}{(s+2)^2+25}$$

$$\mathcal{L}^{-1}\left\{\frac{15}{5}\,\frac{5}{(s+2)^2+25}\right\}$$

$$\Rightarrow \frac{15}{5}\,\mathcal{L}^{-1}\left\{\frac{\overset{\omega}{5}}{(s+\underset{-a}{2})^2+\underset{\omega^2}{25}}\right\}$$

constants pull out

$$= 3\cdot e^{\overset{a}{-2}t}\sin \overset{\omega}{5}t$$

Put everything together:

$$\mathcal{L}^{-1}\left\{ \frac{3}{s^4} + \frac{s-5}{(s-5)^2+9} + \frac{15}{s^2+4s+29} \right\}$$

$$= \underline{\underline{\frac{t^3}{2} + e^{5t}\cos 3t + 3e^{-2t}\sin 5t = f(t)}}$$