Lesson 8 Section 6.2

Transform of y' & $\int y\,dt$

If we plan to use Laplace to solve differential equations, we need to know what the Laplace transform of the derivative of a function is:

to do this, we go back to the definition of the Laplace Transform!

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

so if we want

derivative of $f(t)$

$$\mathcal{L}\{f'(t)\} = \underbrace{\int_0^\infty e^{-st} f'(t)\,dt}$$

how do we do this w/o knowing $f(t)$?

Integration by parts:

$$\cancel{\int u\,dv = vdu - \int}$$

$$\int u\,dv = uv - \int v\,du$$

$u = e^{-st}$ $\qquad du = -se^{-st}$

$dv = f'(t)$ $\qquad v = f(t)$

so, $\mathcal{L}\{F'(t)\} = e^{-st}F(t)\Big|_0^\infty - \underbrace{\int_0^\infty -se^{-st}F(t)\,dt}_{\text{does this look familiar?}}$

see growth restriction pg 225

there is some caution here, but we will forgo it

$e^{-\infty}F(\infty) - e^{0}F(0)$

$0 \qquad F(0)$

$-F(0)$

$+ s\underbrace{\int_0^\infty e^{-st}F(t)\,dt}_{\text{how about now?}}$

$s \cdot F(s)$

so,

$$\mathcal{L}\{F'(t)\} = sF(s) - F(0)$$

similarly it can be shown:

$$\mathcal{L}\{F''(t)\} = s^2F(s) - sF(0) - F'(0)$$

&

$$\mathcal{L}\left\{\int_0^t F(\tau)\,d\tau\right\} = \frac{F(s)}{s}$$

Great... so how does this help us w/solving ODEs?

Lets look at one:

$$y'' + 5y' + 4y = 2\cos 4t$$
$$y(0) = 1, \quad y'(0) = -1$$

① Take the Laplace Transform of the entire equation:

$$\mathcal{L}\{y'' + 5y' + 4y = 2\cos 4t\}$$

& due to linearity that gives us:

$$\mathcal{L}\{y''\} + \mathcal{L}\{5y'\} + \mathcal{L}\{4y\} = \mathcal{L}\{2\cos 4t\}$$

$$s^2 Y(s) - s\,\overset{1}{y(0)} - \overset{-1}{y'(0)} + 5s\,Y(s) - 5\,\overset{1}{y(0)} + 4\,Y(s) = \frac{2s}{s^2+16}$$

$$Y(s)\underbrace{(s^2+5s+4)}_{\text{System, exactly the same polynomial as the characteristic equation... coincidence?? I think not!}} \underbrace{-s+1-5}_{\text{from initial conditions}} = \underbrace{\frac{2s}{s^2+16}}_{\text{Forcing Function}}$$

$$Y(s)\,(s^2+5s+4) = \frac{2s}{(s^2+16)} + s + 4$$

Now, solve for $Y(s)$

$$Y(s) = \underbrace{\frac{2s}{(s^2+16)(s^2+5s+4)}}_{\text{Grand finale!}} + \underbrace{\frac{s}{(s^2+5s+4)}}_{\text{piece 1}} + \underbrace{\frac{4}{(s^2+5s+4)}}_{\text{piece 2}}$$

Done! -- Not really, we need to take the inverse transforms -- but first we need to do some Algebra!

1st - Piece 1

$$\underbrace{\frac{s}{s^2+5s+4}}_{\text{can we factor?}} = \underbrace{\frac{s}{(s+4)(s+1)}}_{\text{yes! -- partial fractions now}} = \frac{A}{s+4} + \frac{B}{s+1}$$

$\Rightarrow$ $A = \dfrac{s}{(s+1)}$ when $s = -4$

$A = 4/3$

&

$B = \dfrac{s}{s+4}$ when $s = -1$

$B = -1/3$

$$\mathcal{L}^{-1}\left\{ \frac{4}{3}\cdot\frac{1}{s+4} + -\frac{1}{3}\,\frac{1}{s+1} \right\} = \frac{4}{3}e^{-4t} - \frac{1}{3}e^{-1t}$$

Piece 1

2nd - Piece 2

$$\frac{4}{(s^2+5s+4)} = \frac{A}{s+1} + \frac{B}{s+4}$$

$(s+1)(s+4)$

$A = \frac{4}{(s+4)}$ when $s=-1$

$A = 4/3$

$B = \frac{4}{(s+1)}$ when $s=-4$

$B = -4/3$

$$\mathcal{L}^{-1}\left\{ -\frac{4}{3}\frac{1}{s+4} + \frac{4}{3}\frac{1}{s+1} \right\} = -\frac{4}{3}e^{-4t} + \frac{4}{3}e^{-t}$$

So far we've done piece 1 & 2
combining them we get:
e^{-t}

but, we still have

$\frac{2s}{(s^2+16)(s+1)(s+4)}$ to break up

$$\frac{2s}{(s+1)(s+4)(s^2+16)} = \frac{A}{(s+1)} + \frac{B}{(s+4)} + \frac{Cs+D}{(s^2+16)}$$

use the same techinque

1st Multiply by $(s+1)$, then let $s=-1$, giving us

$$\frac{2(-1)}{(-1+4)(-1^2+16)} = A = \frac{-2}{3 \cdot 17} = \frac{-2}{51}$$

Now Multiply by $(s+4)$, & let $s=-4$

$$\frac{2(-4)}{(-4+1)(-4^2+16)} = B = \frac{-8}{-3 \cdot 32} = \frac{8}{96}$$

Finally, Multiply by (s^2+16) & let $s=4i$

$$\frac{2(4i)}{(4i+1)(4i+4)} = C\cancel{s}^{4i} + D = \frac{8i}{-12+20i} \frac{(-12-20i)}{(-12-20i)} = \frac{160-96i}{144+400}$$

$$C\cancel{s}^{4i} + D = \frac{160-96i}{544}$$ } seperate real & imaginary parts

$$C4i = \frac{-96i}{544}$$

$$D = \frac{160}{544}$$

$$C = \frac{-24}{544}$$

So we have:

$$\mathcal{L}^{-1}\left\{ \frac{-2}{51}\,\frac{1}{s+1} + \frac{8}{96}\,\frac{1}{s+4} - \frac{24}{544}\,\frac{s}{s^2+16} + \frac{160}{544}\cdot\frac{1}{4}\cdot\frac{4}{s^2+16} \right\}$$

$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$

$$\frac{-2}{51}e^{-t} + \frac{8}{96}e^{-4t} + \frac{-24}{544}\cos 4t + \frac{40}{544}\sin 4t$$

Combine everything!

$$\underbrace{e^{-t}}_{\text{from previous two pieces}} - \frac{2}{51}e^{-t} + \frac{8}{96}e^{-4t} - \frac{24}{544}\cos 4t + \frac{40}{544}\sin 4t$$

$$y(t) = \underbrace{\frac{49}{51}e^{-t} + \frac{8}{96}e^{-4t}}_{\text{transient response}} \underbrace{- \frac{24}{544}\cos 4t + \frac{40}{544}\sin 4t}_{\text{Steady state response}}$$

?- Can you tell me if the system is overdamped, underdamped or critically damped from our transient response?

Bonus Points to the First one to email me the correct answer! & Why also!! (Prior to Mondays Class)